On the Parallels Between Minimal Surfaces and Einstein Four-Manifolds

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Abstract

Minimal surfaces and Einstein manifolds are among the most natural structures in differential geometry. Whilst minimal surfaces are well understood, Einstein manifolds remain far less so. This exposition synthesises together a set of parallels between minimal surfaces embedded in an ambient three-manifold, and Einstein four-manifolds. These parallels include variational formulations, topological constraints, monotonicity formulae, compactness and epsilon-regularity theorems, and decompositions such as thick/thin and sheeted/non-sheeted structures.

Though distinct in nature, the striking analogies between them raises a profound question: might there exist circumstances in which these objects are, in essence, manifestations of the same underlying geometry? Drawing on foundational results such as Jensen's theorem, Takahashi's theorem, and a conjecture of Song, this work suggests a bridge between the two structures. In particular, it shows that certain Einstein four-manifolds admit a minimal immersion into a higher-dimensional sphere. A key example of this is the embedding of $\mathbb{CP}^2$ into S^7 via the Veronese map, where it arises as a minimal submanifold.

Keywords Minimal surfaces · Einstein manifolds · Differential geometry · Geometric analysis

1 Introduction

The principal aim of this exposition is to demonstrate that a collection of fascinating parallels exist between Einstein four-manifolds and minimal surfaces embedded in ambient three-manifold. As we shall see, these connections intersect differential geometry, geometric analysis, and topology.

In this introductory chapter, we gently develop the intuition behind each of these structures, so that the reader may fully appreciate the depth and elegance of the similarities.

1.1 Minimal Surfaces

Minimal surfaces are among the oldest and most fundamental objects in differential geometry. For over 250 years, they have fascinated mathematicians from Euler to Lagrange to Plateau. Defined as surfaces that locally minimise area, they arise as critical points of a variational principle which is satisfied precisely when the mean curvature vanishes across the entire surface. Physically, they manifest as soap films formed when a wire contour is dipped into a soapy solution, as demonstrated by an experiment famously conducted by Plateau in the 19th century. This led to the Plateau problem, which seeks the surface of least area spanning a given closed boundary [Plateau, 1873], which remains central to the study of minimal surfaces today.

Definition 1 (Minimal surface)**.** *A surface in $\mathbb{R}^3$ is said to be minimal if its mean curvature vanishes at every point. That is:*

$$H = \frac{1}{2}(k_1 + k_2) = 0 \iff k_1 = -k_2, \tag{1}$$

where k_1 and k_2 denote the principal curvatures of the surface [Callens and Zadpoor, 2018].

This geometric condition represents just one of several equivalent characterisations of minimal surfaces. Analytically, if a surface is expressed as the graph of a function $u(x, y)$, then the condition of vanishing mean curvature yields the minimal surface equation:

$$(1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} + (1 + u_y^2)u_{xx} = 0 \tag{2}$$

From the variational approach, minimal surfaces arise as critical points of the area functional:

$$\mathcal{A} = \int_M \mathrm{d}A, \tag{3}$$

where $\mathrm{d}A$ denotes the induced area element on the surface M.

Classical examples of minimal surfaces include the catenoid, Enneper's surface, Bour's minimal surface, and the gyroid. Such objects arise not only in mathematics, but also in architecture, crystal geometry, and material sciences.

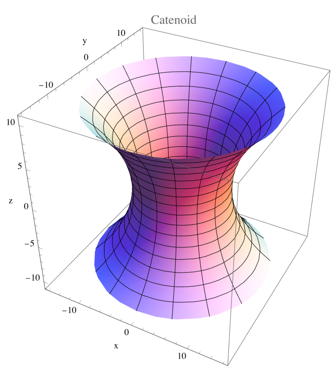

Figure 1: The catenoid

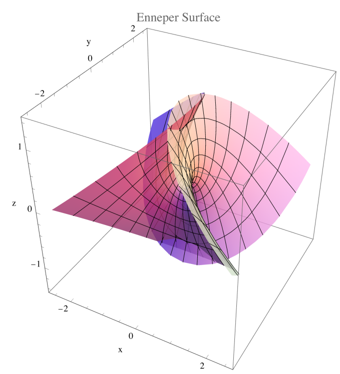

Figure 2: Enneper's surface

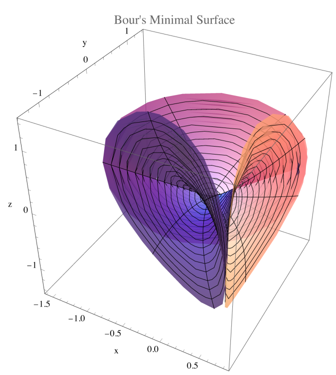

Figure 3: Bour's minimal surface

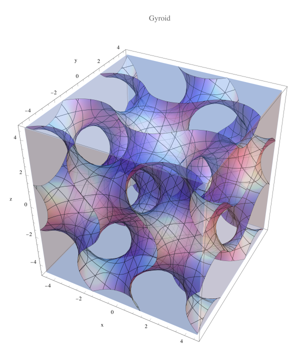

Figure 4: The gyroid (an infinitely connected periodic minimal surface)

Figure 5: Examples of classical minimal surfaces.

1.2 Einstein Manifolds

Einstein manifolds arise naturally within Riemannian geometry, which is the branch of differential geometry where we endow smooth manifolds with a metric structure that allows us to measure curvature, angle, and distance [1]

[1]See [Carmo, 1993] for a full treatment of Riemannian geometry.

An Einstein manifold is a Riemannian manifold whose Ricci curvature tensor is proportional to the metric tensor, offering a natural generalisation of constant curvature. These manifolds gain physical interpretation in general relativity, where the metric serves as a solution to the vacuum Einstein field equations. We present the following definition, taken from [Besse, 1987]:

Definition 2 (Einstein manifold)**.** *A Riemannian manifold (M, g) is said to be Einstein if there exists a constant $\lambda \in \mathbb{R}$ such that:*

$$\mathrm{Ric}_g = \lambda g, \tag{4}$$

where Ric_g denotes the Ricci curvature tensor of g.

Our knowledge of Einstein manifolds can be succinctly organised into dimensional cases.

In dimension two, there is the most elementary manifestation: a Riemannian manifold is Einstein if and only if it has constant Gaussian curvature. These are locally isometric to the sphere, the Euclidean plane, or the hyperbolic plane.

In dimension three, a Riemannian metric is Einstein if and only if it has constant sectional curvature. This requires the manifold's universal cover to be diffeomorphic to either the three-sphere S^3, Euclidean space $\mathbb{R}^3$, or hyperbolic space $\mathbb{H}^3$. This aligns with Thurston's geometrisation conjecture [Thurston, 1982], now a theorem [Grisha, 2002, 2003a,b], which states that every closed three-manifold can be decomposed into pieces admitting one of eight canonical geometric structures.

The four-dimensional case is more challenging, with the most viable tool at our disposal being elimination, whereby one can single out a number of four-manifolds which do not admit Einstein metrics by identifying topological obstructions to their existence. Chief among these obstructions is the Hitchin-Thorpe inequality [Thorpe, 1969, Hitchin, 1974], which asserts that any compact, oriented four-manifold satisfying:

$$|\tau(M)| \leq \frac{2}{3}\chi(M), \tag{5}$$

may admit an Einstein metric, while those violating this inequality certainly do not. Here, $\tau(M)$ is the signature of the manifold, and $\chi(M)$ is the usual Euler characteristic.

Since we have introduced two fundamental objects in differential geometry, it is worthwhile to formally define each [2]:

Definition 3 (Signature of a Manifold)**.** *Let M be a closed, oriented, smooth manifold of dimension $4k$, for $k \in \mathbb{N}$. The signature $\tau(M)$ of M is an integer defined via the intersection form:*

$$Q_M : H^{2k}(M;\mathbb{R}) \times H^{2k}(M;\mathbb{R}) \to \mathbb{R}, \quad (\alpha, \beta) \mapsto (\alpha \smile \beta, [M]), \tag{6}$$

where $\smile$ denotes the cup product, and $[M] \in H_{2k}(M;\mathbb{R})$ is the fundamental class of M.

The signature $\tau(M)$ is the number of positive eigenvalues minus the number of negative eigenvalues of a matrix representing Q_M [Milnor and Stasheff, 1974, Hirzebruch, 1995, Atiyah and Singer, 1968].

Definition 4 (Euler characteristic)**.** *Let M be a smooth manifold with a finite CW-complex structure, with c_k the number of k-cells. The* Euler characteristic *$\chi(M)$ is given by*

$$\chi(M) = \sum_{k=0}^{\dim M} (-1)^k c_k. \tag{7}$$

[Hatcher, 2002]

Examples of known four-dimensional Einstein manifolds include the 4-sphere S^4, the complex projective plane $\mathbb{CP}^2$ (with the Fubini-Study metric 13), the K3 surface with a Ricci-flat Kähler metric, and the Eguchi-Hanson metric.

In higher dimensions, the knowledge of topological restrictions to the existence of Einstein manifold is more murkey. Many manifolds with dimension greater than four admit a negative Einstein metric [Besse, 1987]. When a positive Einstein metric is required, existence hinges on two more delicate criteria. Myers' Theorem [Myers, 1941] states that if a complete Riemannian manifold has Ricci curvature bounded below by a positive constant, then its fundamental

[2] The recommendation is that the reader consults [Hatcher, 2002] for a more thorough understanding, as these definitions rely on the fundamentals of algebraic topology.

group must be finite and its diameter must be bounded. Thus, any manifold admitting a positive Einstein metric must also have finite fundamental group, immediately excluding many topological types. The scalar curvature also plays a crucial role, often investigated via the Yamabe problem [Yamabe, 1960], which considers whether a conformal class contains a metric of constant positive scalar curvature. Examples of Einstein manifolds in these higher dimensions is the Calabi-Yau manifolds (an important manifold in string theory), a quintic 3-fold in $\mathbb{CP}^2$, and G_2 manifolds.

2 The Parallels Between Minimal Surfaces and Einstein Four-Manifolds

The purpose of this chapter is to explore a number of parallels between minimal surfaces embedded within a three-manifold, and Einstein four-manifolds. These analogies were noted in [Song, 2021], building upon several prior developments in geometry and analysis. We attempt to articulate each parallel as clearly as possible, providing formal definitions, proofs where necessary, and referencing the original papers, so that the interested reader may investigate each idea in more depth, should they so wish.

Let us note that at no point do we claim that these objects are the same, in fact they are distinct and live in different spaces. However, this distinction makes the parallels even more intriguing.

2.1 Variational Structures

We begin with the most tempting parallel: both minimal surfaces and Einstein metrics arise as critical points of fundamental variational principles.

Minimal surfaces are critical points of the area functional:

$$\mathcal{A} = \int_M \mathrm{d}A, \tag{8}$$

where M is a surface in a Riemannian manifold and $\mathrm{d}A$ is the induced area element.

Analogously, Einstein four-metrics are critical points of the Einstein-Hilbert functional, which integrates scalar curvature over the manifold:

$$\mathcal{E}(g) = \frac{\int_M R_g \,\mathrm{vol}_g}{(\mathrm{Vol}(M,g))^{1/2}}, \tag{9}$$

where R_g is the scalar curvature associated to the metric g, and dVol_g is the Riemannian volume form.

This variational similarity deepens if we consider the geometric flows naturally associated with each structure.

2.1.1 Mean Curvature Flow

Mean Curvature Flow (MCF) is a geometric evolution equation in which a hypersurface deforms over time in the direction of steepest descent for area. The flow tends to smooth irregularities, but singularities can develop where the hypersurface pinches off, reflecting changes in the topology.

Definition 5 (Mean Curvature Flow). *Let $M_t \subset \mathbb{R}^3$ be a family of smoothly embedded hypersurfaces depending on time t. The mean curvature flow is the evolution equation:*

$$\frac{\partial X}{\partial t} = -H\nu, \tag{10}$$

where:

- $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ *is a smooth family of immersions*
- H *is the mean curvature*
- ν *is the unit normal vector to the hypersurface*

[Nunemacher, 2003].

Minimal surfaces correspond to fixed points of this flow.

2.1.2 Ricci Flow

The Ricci flow, introduced by Hamilton in 1982, serves as the intrinsic analogue of MCF for Riemannian metrics. It deforms a Riemannian metric in the direction of steepest descent for total scalar curvature under appropriate constraints on the volume, smoothing out curvature irregularities.

Definition 6 (Ricci Flow)**.** *Let M be a Riemannian manifold, and $g(t)$ be a one-parameter family of Riemannian metrics on M. The Ricci flow is the geometric evolution equation:*

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g(t)), \tag{11}$$

where $\mathrm{Ric}(g)$ is the Ricci curvature tensor of the evolving metric.

[Chow and Knopf, 2011, Brendle, 2010]

Under suitable conditions and normalisation, Einstein metrics can arise as fixed points of this flow.

Both flows share an underlying philosophy: evolve the geometric structure (extrinsically in the case of mean curvature, and intrinsically in the case of Ricci curvature) toward a state of balance. In both settings, the development of singularities not only signals the breakdown of smooth evolution, but often reveals deep topological features of the underlying space.

2.2 Second Variation

We now turn to the second variation of each geometric object, investigating how the respective functionals behave under second-order deformations. This deepens the narrative developed so far: not only do minimal surfaces and Einstein metrics arise as critical points of variational principles, but their stability is goverened by elliptic operators whose spectra encode geometric information, leading to the notion of Morse index.

2.2.1 Minimal Surfaces

Let $f : M \to N$ be a minimal immersion of a compact surface M into a Riemannian manifold N. Consider a normal variation $V \in \Gamma(\nu(M))$, where $\nu(M)$ denotes the normal bundle of the immersion and $\Gamma(\nu(M))$ the space of its smooth sections. Then the second variation of the area functional takes the form:

$$\delta^2 \mathcal{A}(V,V) = \int_M \langle \mathcal{J}V, V\rangle \,\mathrm{d}\mu. \tag{12}$$

Let us briefly define $\mathcal{J}$, which is the Jacobi operator.

Definition 7 (Jacobi Operator)**.** *The Jacobi operator is a second-order, elliptic, self-adjoint differential operator acting on normal vector fields. Its general form is:*

$$\mathcal{J} = \Delta^{\perp} + |A|^2 + \mathrm{Ric}_N(\nu,\nu), \tag{13}$$

where $\Delta^{\perp}$ is the Laplace-Beltrami operator on the normal bundle, A is the second fundamental form of the immersion, and $\mathrm{Ric}_N(\nu,\nu)$ is the Ricci curvature of the ambient manifold N in the direction of the unit normal vector field ν [Fischer-Colbrie and Schoen, 1980].

The Morse index of the minimal surface is defined as the number of negative eigenvalues of $\mathcal{J}$. It measures the maximal dimension of the subspace of deformations along which the area decreases (the space of unstable directions in the variational landscape) [3].

2.2.2 Einstein Metrics and the Einstein Index

Let (M^4, g) be a compact Einstein four-manifold. A variation of the metric is given by a symmetric $(0,2)$-tensor $h \in S^2(T^*M)$. To isolate meaningful variations, we restrict attention to transverse-traceless tensors:

[3] This is a simplification of what is a rich field, so see [Milnor and Weaver, 1997] for more information.

$$\mathrm{TT}_g = \left\{ h \in S^2(T^*M) \mid \mathrm{Tr}_g(h) = 0,\ \delta_g h = 0 \right\}, \tag{14}$$

which preserve volume and are orthogonal to conformal transformations.

The second variation of the Einstein-Hilbert functional, restricted to TT_g, is governed by the Lichnerowicz Laplacian [Galloway, 1988]:

$$\Delta_L h = \nabla^* \nabla h + 2\mathring{R}h, \tag{15}$$

where $\nabla^*\nabla$ is the rough Laplacian, and $\mathring{R}$ is the curvature operator acting on symmetric 2-tensors, defined locally by:

$$(\mathring{R}h)_{ij} = R_{ikjl} h^{kl}. \tag{16}$$

The rough Laplacian will appear again later, warranting a formal definition.

Definition 8 (Rough Laplacian). *Let ∇_X denote the covariant differentiation along the vector field X. For a smooth section $\phi \in \Gamma(E)$, , where E is a vector bundle over M, the rough Laplacian is defined as:*

$$\Delta\phi = -\,\mathrm{Tr}_g(\nabla^2\phi) = -\sum_{i=1}^{n} \left(\nabla_{e_i}\nabla_{e_i}\phi - \nabla_{\nabla_{e_i} e_i}\phi \right), \tag{17}$$

where $\{e_i\}$ is a local orthonormal frame. The operator Δ is second-order, elliptic, and self-adjoint.

The index of the Einstein metric is defined as the number of negative eigenvalues of Δ_L acting on TT_g. It quantifies the number of volume-preserving deformations that decrease the Einstein–Hilbert functional to second order.

In both settings, the second variation demonstrates a spectral structure intimately tied to the underlying geometry, nodding towards a deeper variational landscape. The Jacobi operator and Lichnerowicz Laplacian reveal how curvature controls stability, and their spectra bridge analytic, geometric, and topological domains. The Morse index, in either context, measures how ‘saddle-like’ a configuration is within its functional landscape. These parallels suggest that not only the existence, but also the stability of these critical geometric objects is governed by analogous principles.

3 Monotonicity and Local Rigidity

Both minimal surfaces and Einstein manifolds obey natural monotonicity formulae, which constrain the behaviour of geometric quantities under rescaling. These constraints are not mere technicalities, instead encoding curvature information and guiding our understanding of local-to-global behaviour in geometric analysis.

3.1 Minimal Surfaces

Let $\Sigma \subset N$ be a minimal surface immersed in a Riemannian 3-manifold N. For any point $p \in \Sigma$ and sufficiently small $r > 0$, the function:

$$r \mapsto \frac{\mathrm{Area}(\Sigma \cap B(p,r))}{\pi r^2}, \tag{18}$$

is monotone non-decreasing up to curvature correction terms when the ambient space N is not flat. In the Euclidean case $N = \mathbb{R}^3$, the function is strictly monotonic and becomes constant if and only if Σ is a plane near p [Meeks and Pérez, 2012, Brendle, 2023, White, 2016].

This reflects the second-order minimality condition satisfied by Σ, where the rate of area growth relative to r^2 near a point quantifies how close the surface is to being flat. Thus, it serves as a regularity tool and as a control on local geometry.

3.2 Einstein Manifolds

Let (M^4, g) be an Einstein manifold. Then for any $p \in M$ and sufficiently small $r > 0$, the normalised volume function:

$$r \mapsto \frac{\mathrm{Vol}(B(p,r))}{r^4}, \tag{19}$$

is almost monotone non-increasing as $r \to 0$, with equality if and only if the metric is locally flat at p. This local volume comparison reflects the scalar curvature constraints of the Einstein condition and arises from the Bishop-Gromov volume comparison theorem, which is a foundational result in comparison geometry.

In this context, deviation from constancy in the normalised volume function measures how the Einstein condition influences volume growth. The result is thus not only a local geometric diagostic, but is also a powerful tool in compactness and convergence theory, particularly in the Cheeger-Gromov framework for collapsing and compactness.

The presence of monotonicity formulae in both settings illustrates a deeper structural similarity, with each expressing a local rigidity. Both formulae become equality statements in flat geometry and deviation from monotonicity measures curvature in a quantifiable way. These monotonicity restrictions thus play a dual role: they regulate geometry and hint at underlying analytical principles governing stability, regularity, and convergence in geometric flows.

4 Epsilon-Regularity and Energy Thresholds

Epsilon-regularity is an incredibly useful tool in geometric analysis, particularly in the study of geometrical partial differential equations. Though this tool represents a concept which is rich in analytic formalism, the idea can be elegantly summarised: if one assumes some weak control on the solution for the equation in question through analytic tools, then the assumed control can be used to prove a stronger control of the system. That is, when the total curvature in a region is small enough, one can deduce precise control on the pointwise geometry [4].

4.1 Minimal Surfaces

For minimal surfaces, a classical result due to Choi and Schoen proposes that an embedded minimal surface in a Riemannian 3-manifold remains regular where the total curvature is sufficiently small, which prevents in regions where the total curvature is sufficiently small. Below a critical threshold, singularities are ruled out [Haslhofer, 2024].

Theorem 1 (Choi-Schoen)**.** *There exist constants $\varepsilon > 0$ and $C > 0$ such that the following holds. Let Σ be an embedded minimal surface in a Riemannian 3-manifold N, and let $p \in \Sigma$. If $r > 0$ is sufficiently small and:*

$$\int_{\Sigma \cap B(p,r)} |A|^2 < \varepsilon, \tag{20}$$

then:

$$\sup_{\Sigma \cap B(p,r/2)} |A|^2 \leq \frac{C}{r^2}, \tag{21}$$

where $|A|$ denotes the norm of the second fundamental form of Σ [Choi and Schoen, 1985].

In other words, if a minimal surface does not exhibit excessive curvature in a small region, then we can guarantee that its curvature remains controlled in an even smaller concentric ball. This provides an essential tool in regularity and compactness theories for families of minimal surfaces, as well as underpinning several blow-up results.

Proof. We outline the proof of Choi-Schoen's Theorem using a classical bootstrapping technique, starting from small L^2-control on the second fundamental form, and deriving a pointwise curvature bound.

Let $\Sigma \subset N^3$ be an embedded minimal surface. The second fundamental form A satisfies the Simons inequality:

[4] The reader is recommended to consult [Tao, 2009] for a more thorough explaination of this

$$\tfrac{1}{2}\Delta|A|^2 \geq |\nabla A|^2 - C_1|A|^4 - C_2|A|^2,$$

where C_1, C_2 depend only on a bound for $\|\mathrm{Rm}_N\|_{C^1}$ in the chosen coordinate patch
Let $\phi \in C_c^\infty(B_\Sigma(p,r))$ be a smooth cut-off function satisfying:

$$0 \leq \phi \leq 1, \quad \phi \equiv 1 \text{ on } B(p, r/2), \quad |\nabla\phi| \leq \frac{2}{r}.$$

Multiply Simons inequality by ϕ^2 and integrate over Σ:

$$\int_\Sigma \phi^2\Delta|A|^2 \geq \int_\Sigma \left(2\phi^2|\nabla A|^2 - 2C_1\phi^2|A|^4 - 2C_2\phi^2|A|^2\right),$$

while integration by parts yields: $$\int_\Sigma \phi^2\Delta|A|^2 = -\int_\Sigma \langle\nabla|A|^2, \nabla(\phi^2)\rangle \leq \int_\Sigma 4\phi|\nabla\phi||A||\nabla A|.$$

Applying the Cauchy-Schwarz and Young inequalities gives:

$$4\phi|\nabla\phi||A||\nabla A| \leq \tfrac{1}{2}\phi^2|\nabla A|^2 + \tfrac{8}{r^2}|A|^2.$$

Combining these estimates:

$$\int_\Sigma \phi^2|\nabla A|^2 + C_1\int_\Sigma \phi^2|A|^4 \leq \frac{C}{r^2}\int_{\Sigma\cap B(p,r)} |A|^2.$$

Note that in particular:

$$\int_{\Sigma\cap B(p,r)} |A|^4 \leq \frac{C}{r^2}\int_{\Sigma\cap B(p,r)} |A|^2.$$

The final step is to apply a Moser-type iteration argument, which is a bootrastrapping process for elliptic PDEs. By the Michael-Simon Sobolev inequality on minimal surfaces:

$$\left(\int (\phi|A|)^4\right)^{1/2} \leq C_S \int |\nabla(\phi|A|)|^2.$$

With the previous L^4-bound, one obtains a recursive estimate:

$$\||A|\|_{L^{2p}(B(p,r/2))} \leq \frac{C_p}{r^{\alpha_p}}\||A|\|_{L^p(B(p,r))}, \quad p = 2, 4, 8, \ldots$$

With Moser-type interation, we send $p \to \infty$ and deduce the desired bound:

$$\sup_{B(p,r/2)} |A|^2 \leq \frac{C}{r^2}\int_{B(p,r)} |A|^2.$$

Finally, choosing $\varepsilon < 1/C$, one obtains:

$$\sup_{B(p,r/2)} |A|^2 < \frac{1}{r^2},$$

which implies the claimed pointwise estimate:

$$\sup_{B(p,r/2)} |A|^2 \leq Cr^{-2}.$$

[Chen, 1984, Leon, 1983, Brendle, 2023]

□

4.2 Einstein Manifolds

An analogous epsilon-regularity result holds for Einstein four-manifolds. In foundational work by Anderson, and then subsequent contributions by Cheeger-Tian, Nakajima, and Gao, it was shown that pointwise curvature bounds can be derived from the integral $L^{n/2}$-control of the Riemann curvature tensor on small balls. In dimension four, this condition becomes an L^2-bound.

Theorem 2 (Anderson, Cheeger–Tian, Nakajima, Gao)**.** *There exist constants $\varepsilon > 0$ and $C > 0$ such that the following holds. Let (M^4, g) be an Einstein manifold, and let $p \in M$. If $r > 0$ is sufficiently small and:*

$$\int_{B(p,r)} |\mathrm{Rm}|^2 < \varepsilon, \tag{22}$$

then:

$$\sup_{B(p,r/2)} |\mathrm{Rm}| \leq \frac{C}{r^2}, \tag{23}$$

where $|Rm|$ denotes the pointwise norm of the Riemann curvature tensor. [Anderson, 1989, Cheeger p and Tian, 2006, Nakajima, 1988, Gao, 1990]

This theorem states that if the L^2-norm of the total curvature is sufficiently small in a ball $B(p, r)$, then the curvature remains uniformly bounded on the smaller concentric ball $B(p, r/2)$, proving control over the geometry of Einstein manifolds under integral curvature bounds.

Sketch. Following the method used for proving the Choi-Schoen theorem, we again follow the standard bootstrap argument, moving from integral to pointwise control.

A key step is the use of harmonic coordinates. Anderson [Anderson, 1989] showed that if $\int_{B(p,r)} |\mathrm{Rm}|^2 < \varepsilon$ for sufficiently small ε, then there exist harmonic coordinates $\{x_i\}$ on $B(p, r)$ such that the metric satisfies:

$$\|g_{ij} - \delta_{ij}\|_{C^{1,\alpha}(B(p,r))} \leq \tfrac{1}{2}, \qquad \Lambda^{-1}\delta_{ij} \leq g_{ij} \leq \Lambda\delta_{ij}, \tag{24}$$

for a uniform $\Lambda > 1$. In these coordinates, the Laplace–Beltrami operator is uniformly elliptic with controlled constants.

On an Einstein manifold, the Riemann curvature tensor satisfies an elliptic equation of the form:

$$\Delta\mathrm{Rm} = \mathrm{Rm} * \mathrm{Rm}, \tag{25}$$

where Δ is the rough Laplacian and $*$ denotes bilinear contractions of curvature terms [5] This identity follows from the second Bianchi identity together with the Einstein condition $\mathrm{Ric} = \lambda g$.

The equation above constitutes a non-linear elliptic system for Rm, with quadratic lower-order terms. Since the equation is elliptic and the background metric is well-controlled in harmonic coordinates, classical elliptic regularity theory, such as Moser iteration or interior L^p-to-L^∞ estimates, can be applied [Bando et al., 1989, Gilbarg and Trudinger, 2001]. This gives:

$$\sup_{B(p,r/2)} |\mathrm{Rm}| \leq \frac{C}{r^2} \left(\int_{B(p,r)} |\mathrm{Rm}|^2 \right)^{1/2}. \tag{26}$$

If the integral is smaller than a universal threshold ε, we may absorb the constant into the right-hand-side to reach the desired estimate.

□

[5] See [Besse, 1987] for more details.

5 Compactness and Convergence

In studying sequences of minimal surfaces embedded in a three-manifold, or Einstein metrics on four-manifolds, one finds that limiting spaces retain smoothness in regions of bounded curvature, but may develop singularities in higher-energy zones. Compactness theorems ensure that under appropriate geometric controls, such degeneration is mild and structured, rather than chaotic.

5.1 Minimal Surfaces

The compactness theory for minimal surfaces has been developed in increasingly refined ways. An important result due to Sharp, Chodosh, Ketover, and Maximo shows that under uniform area and Morse index bounds, a sequence of minimal surfaces converges to a smooth limiting surface, with their topological type controlled.

Theorem 3 (Sharp-Chodosh-Ketover-Maximo)**.** *Let $\{\Sigma_i\}$ be a sequence of closed, embedded minimal surfaces in a Riemannian 3-manifold (N, g), such that:*

$$\mathrm{Area}(\Sigma_i) \leq C, \qquad \mathrm{index}(\Sigma_i) \leq C. \tag{27}$$

Then, after passing to a subsequence, $\Sigma_i \to \Sigma_\infty \subset N$ smoothly on compact subsets. Furthermore, the genus of Σ_i is uniformly bounded, and the limit Σ_∞ is a smooth, embedded minimal surface. [Chodosh et al., 2017]

The interpretation of this is that if a sequence of minimal surfaces has controlled area and genus, then they cannot degenerate too wildly, and instead must settle down to a smooth limiting surface. This result is widely used in variational problems, particularly in studying minimal surfaces embedded in three-manifolds with positive Ricci curvature.

Proof. Let us begin by proving that the index bound will imply the existence of at most finitely many unstable regions. By definition, $\mathrm{index}(\Sigma_i) \leq C$ implies that there are at most C linearly independent directions along which the second variation is negative. Equivalently, one can find at most C disjoint geodesic balls in Σ_i on which the surface is unstable. Outside of these geodesic balls, the surface is unstable [6].

On any stable minimal surface in a 3-manifold, Schoen's curvature estimates give a pointwise bound:

$$|A|^2(p) \leq \frac{C}{\mathrm{dist}(p, \partial B)^2}, \tag{28}$$

for intrinsic balls $B \subset \Sigma_i$ disjoint from $\partial \Sigma_i$. In particular, on compact susbets away from the unstable regions, one obtains a uniform curvature bound:

$$\sup_{\Sigma_i \setminus \bigcup B_r(x_{i,k})} |A| \leq C. \tag{29}$$

We cover each Σ_i by finitely many intrinsic balls of radius $r > 0$, chosen to be sufficiently small so that no ball contains more than one unstable center. On each stable ball, our previous estimate gives:

$$\sup |A| \leq C(r). \tag{30}$$

Since there are at most C unstable balls, we conclude a uniform curvature bound outside of their union:

$$|A|_{\Sigma_i}(x) \leq C', \quad \forall x \in \Sigma_i \setminus \bigcup_{k=1}^{C} B_r(x_{i,k}), \tag{31}$$

where $x_{i,k}$ denotes the centres of the unstable regions.

Let us now see that there is smooth subsequential convergence away from points of issue. The uniform curvature bound on $\Sigma_i \setminus \bigcup B_r(x_{i,k})$, together with the area bound, allow us to apply Allard's compactness theorem to extract a subsequence converging in C^∞ on compact subsets of:

[6] See [Chodosh et al., 2017] for the local picture of degenerations of bounded-index hypersurfaces

$$N \setminus \{\text{at most } C \text{ points}\}. \tag{32}$$

The limit is a smooth minimal lamination. However, the area bound and embeddedness prevent multiplicity and ensure that the convergence is to a single embedded minimal surface away from the concentration points.

In order to control the topology, we use the Gauss-Bonnet theorem to estimate the Euler characteristic:

$$2\pi\chi(\Sigma_i) = \int_{\Sigma_i} K_{\Sigma_i}\, dA = \int_{\Sigma_i} \left(\mathrm{Sec}_N(\tau) - \tfrac{1}{2}|A|^2\right)\, dA, \tag{33}$$

where $\mathrm{Sec}_N(\tau)$ is the ambient sectional curvature in the direction of the tangent plane. Since Sec_N is bounded and $\int |A|^2 dA$ is controlled by $\mathrm{Area} \cdot \sup |A|^2$, this gives a uniform bound on $\chi(\Sigma_i)$, and hence on the genus.

□

5.2 Einstein Four-Manifolds

A similar philosophy holds for sequences of Einstein four-manifolds. Building on work by Anderson, Bando, Nakajima, Gao, Cheeger, and Tian, one finds that a sequence of Einstein manifolds with uniform topological and geometric bounds converges, modulo singularities, to a smooth limiting Einstein orbifold.

Theorem 4 (Anderson-Bando-Nakajima-Gao)**.** *Let $\{(M_i, g_i)\}$ be a sequence of Einstein four-manifolds satisfying:*

$$\chi(M_i) \le C, \qquad \mathrm{Vol}(M_i, g_i) \ge C^{-1}, \qquad \mathrm{diam}(M_i, g_i) \le C. \tag{34}$$

Then after passing to a subsequence, $(M_i, g_i) \to (X, g_\infty)$ in the Gromov–Hausdorff sense, where X is a smooth Einstein orbifold. Furthermore, the number of diffeomorphism types of M_i is finite. [Anderson, 1989, Bando et al., 1989, Gao, 1990]

The interpretation of this theorem closely parallels that for minimal surfaces. Under uniform control of volume, diameter, and topological invariants, the sequence cannot collapse arbitrarily. Instead, it converges to a space that is smooth away from finitely many singular points, where the geometry exhibits cone-like behaviour (modelled on quotient singularities). These compactness results are foundational in moduli space theory and essential for understanding the global geometry of Einstein manifolds.

Proof. Fix a small ε and a volume threshold $v_0 > 0$. For each $p \in M_i$, define the curvature scale:

$$r_\varepsilon(p) = \sup \left\{ r \le 1 \;\middle|\; \int_{B(p,r)} |\mathrm{Rm}|^2 \le \varepsilon \right\}. \tag{35}$$

Decompose the manifold in the following way:

$$M_i^{>} = \left\{ p \in M_i \,\middle|\, \mathrm{Vol}(B(p, r_\varepsilon(p))) > v_0 r_\varepsilon(p)^4 \right\}, \qquad M_i^{\le} = M_i \setminus M_i^{>}. \tag{36}$$

Here, $M_i^{>}$ represents the thick region, where volume at scale r_ε is non–collapsing, and $M_i^{\le}$ is the thin region (note, thick/thin decomposition will be officialy introduced in the next section, as it represents its own parallel).

By ε-regularity, every $p \in M_i^{>}$ lies in a ball on which $|\,\mathrm{Rm}\,|$ is uniformly bounded. Coupling this with the global diameter bound $\mathrm{diam}(M_i) \le C$, and the global non-collapsing $\mathrm{Vol}(B(p,1)) \ge C^{-1}$, this implies higher-order control via usual elliptic regularity.

By Cheeger-Gromov-Hamilton compactness, a subsequence converges smoothly (with multiplicity one) on $M_i^{>}$:

$$(M_i^{>}, g_i) \longrightarrow (X_{\mathrm{reg}}, g_\infty), \tag{37}$$

where X_{reg} is an open dense subset of the limiting Einstein orbifold.

On the thin region $M_i^{\leq}$, the geometry collapses along small-volume balls. The volume collapse at scale ε forces the pointed rescalings $\left(B(p, r_\varepsilon(p)), r_\varepsilon(p)^{-2} g_i\right)$ to converge to a complete Ricci-flat ALE space $\mathbb{R}^4/\Gamma$. Each of these limits contributes an isolated orbifold singularity of type $\mathbb{R}^4/\Gamma$.

Gluing the smooth convergence on $M_i^{>}$ with the orbifold asymptotics of $M_i^{\leq}$ yields:

$$(M_i, g_i) \longrightarrow (X, g_\infty),$$

where X is a smooth Einstein orbifold with finitely many singular points.

The final step is an appeal to an orbifold finiteness theorem (originally due to Anderson-Cheeger, and then refined by Bando-Kasue-Nakajima): *Given uniform bounds on* $\chi(M_i)$*,* $\mathrm{Vol}(M_i)$*, and* $\mathrm{diam}(M_i)$*, there are only finitely many homeomorphism types in the smooth pre-limit sequence.* That is, the possible arrangements of finitely many orbifold singular points and the discrete data of their local groups $\Gamma \subset O(4)$ admits only finitely many combinations under these constraints.

□

6 Thick/Thin and Sheeted/Non-Sheeted Decomposition

In geometric analysis, it is often fruitful to decompose a complicated space into two regions: a ‘thick’ or ‘regular’ part where geometry is well-controlled, and a ‘thin’ or ‘singular’ part where degeneration may occur. This dichotomy appears both in the theory of Einstein four-manifolds and in the study of minimal surfaces, with such processes named thick/thin decomposition and sheeted/non-sheeted decomposition respectively.

6.1 Einstein 4-Manifolds: Thick/Think Decomposition

In the setting of Einstein four-manifolds, the thick/thin decomposition is based on local pointwise control of curvature energy and volume ratios. Fix a small constant $\varepsilon > 0$, determined by epsilon-regularity results, to serve as a threshold for acceptable curvature concentration. At each $p \in M$, define the regularity scale $r_{\varepsilon(p)}$ as the largest radius $r \leq 1$ for which:

$$\int_{B(p,r)} |Rm|^2 \leq \varepsilon. \tag{38}$$

We then look at the volume growth of this ball to see if it is comparable to Euclidean space, or if it collapses. This classifies the point p into one of two regions:

- If $\mathrm{Vol}(B(p, r_\varepsilon(p))) > V_0 \cdot r_\varepsilon(p)^4$, then p lies in the thick region.
- Otherwise, p lies in the thin region, where volume collapses and curvature may concentrate.

This process filters the manifold into thick regions, where we can apply regularity theorems and use comparison geometry, and thin regions, which are far more subtle and curvature might spike or space pinch. The benefit is that we have now isolated those difficult regions.

This process is formalised via the following:

Definition 9 (Thick/thin decomposition for Einstein 4-manifolds)**.** *Let* (M^4, g) *be an Einstein manifold. For fixed constants* $\varepsilon > 0$ *and* $V_0 > 0$*, define:*

$$r_\varepsilon(p) := \sup \left\{ r \in (0,1] : \int_{B(p,r)} |Rm|^2 \leq \varepsilon \right\}, \tag{39}$$

$$M_{>V_0} := \left\{ x \in M : \mathrm{Vol}(B(x, r_\varepsilon(x))) > V_0 \cdot r_\varepsilon(x)^4 \right\}, \tag{40}$$

$$M_{\leq V_0} := \left\{ x \in M : \mathrm{Vol}(B(x, r_\varepsilon(x))) \leq V_0 \cdot r_\varepsilon(x)^4 \right\}. \tag{41}$$

6.2 Minimal Surfaces: Sheeted/Non-Sheeted Decomposition

For minimal surfaces, an analogous decomposition exists, which is used to distinguish regions of high and low area concentration at small scales. Following the framework introduced by Song [Song, 2022], and used in index-based compactness theorems, we fix a small scale $r > 0$ and a threshold $n_0 > 0$. At each point $p \in \Sigma$, define $s(p) \leq r$ to be the largest radius such that the portion $\Sigma \cap B(p, s(p))$ is stable. Then we classify points as follows:

- If $\mathrm{Area}(\Sigma \cap B(p, s(p))) > n_0 \cdot s(p)^2$, we say p lies in the non-sheeted region, indicating possible bubbling or curvature concentration.
- If $\mathrm{Area}(\Sigma \cap B(p, s(p))) \leq n_0 \cdot s(p)^2$, then p lies in the sheeted region, where the surface behaves regularly.

Intuitively, the sheeted region is like a surface composed of well-separated layers, or 'sheets,' with predictable behaviour. In contrast, the non-sheeted region may exhibit folding, bunching, or bubbling, signaling geometric degeneration.

Definition 10 (Sheeted/Non-Sheeted Decomposition for Minimal Surfaces). *Let $\Sigma \subset (N^3, g)$ be a minimal surface. Fix constants $r > 0$ and $n_0 > 0$. Define:*

$$s(p) := \sup\left\{\tilde{r} \leq r : \Sigma \cap B(p, \tilde{r}) \textit{ is stable}\right\}, \tag{42}$$

$$\Sigma_{>n_0} := \left\{x \in \Sigma : \mathrm{Area}(\Sigma \cap B(x, s(x))) > n_0 \cdot s(x)^2\right\}, \tag{43}$$

$$\Sigma_{\leq n_0} := \left\{x \in \Sigma : \mathrm{Area}(\Sigma \cap B(x, s(x))) \leq n_0 \cdot s(x)^2\right\}. \tag{44}$$

6.3 Conclusion

Though these two decompositions arise in very different settings, they share a common philosophical role. Both seek to isolate regions of geometric regularity from those of potential degeneration. In the thick or sheeted regions, the geometry behaves predictably, and powerful compactness theorems apply. In the thin or non-sheeted regions, by contrast, curvature concentrates, topological change can occur, and delicate analysis is required.

7 Reflections on the Parallels and Directions for Further Study

The parallels proposed span the fields of topology, differential geometry, and geometric analysis. The fact that these transcend many areas of mathematics suggests that there may be a deeper conceptual link between the objects which is worth investigating. Some potential research problems which may warrant further study include:

- Characterising the class of four-dimensional Einstein manifolds that admit isometric minimal embeddings into higher-dimensional spaces. This may offer a geometric framework extending the constructions of Hitchin and LeBrun on twistor spaces and self-duality.
- Analyse whether these structural similarities carry over to other elliptical system, such as Yang–Mills fields or metrics of constant scalar curvature. These may all be viewed as moment-map equations, and understanding this shared framework could allow techniques, such as existence or regularity results, to be transferred across systems.
- Explore analogies between the complexification of Einstein geometry (for example, through complexified holonomy or twistor methods) and the classical complexification of minimal surfaces via the Weierstrass–Enneper representation. Are there deeper ties between holomorphic data and the underlying real geometric structure?
- Compare Plateau and free-boundary problems for minimal surfaces with conformal, Robin, and Bartnik-type boundary conditions for Einstein metrics, with the aim of isolating regimes where well-posedness and regularity theories align.

Each of these questions present a compelling avenue for deeper mathematical investigation. In the remainder of this work, we choose to focus on the first question: establishing criteria under which an Einstein four-manifold may be realised as a minimal submanifold in a suitable higher-dimensional ambient space, constructing an explicit example to demonstrate this.

8 Embedding Einstein Four-Manifolds as Minimal Submanifolds

This chapter investigates a concrete connection between Einstein geometry and the theory of minimal submanifolds. Specifically, we consider conditions under which an Einstein four-manifold may be isometrically immersed as a minimal submanifold in a higher-dimensional ambient space. This framework allows us to study Einstein manifolds using techniques from minimal surface theory, and provides a geometric realisation of the parallels explored throughout this exposition.

To prepare the ground, we begin by examining a classification result due to Jensen, which shows that locally homogeneous Einstein four-manifolds are necessarily locally symmetric. This remarkable theorem reduces our subsequent embedding problem to the realm of locally symmetric spaces.

8.1 Jensen's Theorem

The defining feature of an Einstein manifold is constant Ricci curvature, which is a strong constraint to place on a group of objects. Jensen discovered that in four dimensions, all locally homogeneous Einstein four-manifolds are locally symmetric, which imposes an even stronger constraint on the system.

Definition 11 (Locally Symmetric Space)**.** *A Riemannian manifold (M, g) is locally symmetric if its Riemann curvature tensor is parallel:*

$$\nabla R = 0 \tag{45}$$

Theorem 5 (Jensen's Theorem)**.** *Every locally homogeneous Riemannian Einstein four-manifold is locally symmetric [Jensen, 1969].*

Jensen's original proof proceeded by a brute-force classification of four-dimensional Lie algebras. We choose instead to follow the curvature-tensor approach of [Derdzinski, 2000], which is more streamlined.

Proof. Let (M, g) be a locally homogeneous Einstein four-manifold. The idea that is that we will first prove the curvature operator has constant eigenvalues; consequently the eigenvalues of the self-dual and anti-self-dual Weyl operators $W^{\pm}$ are constant, and then to finally arrive at $\nabla R = 0$ by Ambrose-Singer.

Let us begin with the following lemma:

Lemma 1. *Let (M^4, g) be Einstein. If either W^+ or W^- has pointwise constant eigenvalues, then the corresponding tensor is parallel: $\nabla W^{\pm} = 0$ [Derdzinski, 2000].*

Proof. Consider W^+ acting on the rank-three bundle Λ^2_+. Since g is an Einstein metric, the trace-free Ricci part of the curvature vanishes, with the decomposition is given by:

$$R = W + \frac{s}{12} g \owedge g.$$

Then all first-order variations of the curvature live in the Weyl tensor W. Since the eigenvalues of W^+ are constant, one can choose a smooth orthonormal basis of Λ^2_+ at each point which diagonalises W^+ smoothly:

$$W^+ = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3), \qquad \lambda_i = \text{const}.$$

Differentiating this relation shows that the derivative ∇W^+ must preserve the eigenspace structure. But we note that in dimension four, any trace-free symmetric 3×3 matrix with distinct eigenvalues has a trivial stabiliser under conjugation in $SO(3)$. Thus, the only way that ∇W^+ can be compatible with a globally fixed diagonalisation is for it to vanish, $\nabla W^+ = 0$. The same argument applies to W^-.

□

This is a streamlined version of what is a calculation-heavy proof, and so the reader is recommended to consult [Derdzinski, 2000], specifically Theorem 7.1, which carries out the index-wise computation using the first Bianchi identity, and the fact that W^+ is trace-free.

Now, we have a corollary to this lemma, which we will likewise prove:

Corollary 1. *If $\nabla W = 0$ on an Einstein four-manifold, then $\nabla R = 0$, and hence (M, g) is locally symmetric.*

Proof. On an Einstein manifold, the scalar curvature s is constant, and so the term $\frac{s}{12} g \owedge g$ is parallel. With $\nabla W = 0$, the decomposition in the previous proof shows that $\nabla R = 0$.

Local symmetry follows from the Ambrose-Singer theorem, which states that a Riemannian manifold with $\nabla R = 0$ is locally symmetric, meaning that there exist local geodesic symmetries about every point which preserve the metric and connection. Thus, (M, g) is locally symmetric, as required.

□

□

Let us appreciate how Jensen's theorem is profound. In dimension four, there is the inevitability that a locally homogeneous Einstein manifold be locally symmetric, a large constraint to place on a system of already highly constrained objects. Not only this, but we have a theorem which shows that symmetry is something that isn't just imposed, but is something that can emerge. We carry this theorem through to the next section, where we link this to minimal submanifolds.

8.2 Minimal Embeddings of Symmetric Spaces

The bridge between Einstein four-manifolds and minimal submanifolds is made possible by a series of classical results in differential geometry.

Theorem 6 (Takahashi, 1966)**.** *Every irreducible compact symmetric space admits a minimal isometric immersion into a Euclidean sphere [Takahashi, 1966].*

Sketch. Takahashi first proved that an isometric an isometric immersion $x : M \to \mathbb{R}^{m+k}$ of a Riemannian manifold M into Euclidean space satisfying $\Delta x = \lambda x$ with $\lambda \neq 0$, is minimal in a sphere of radius $r = \sqrt{m/\lambda}$.

Now, let M be a compact homogeneous Riemannian manifold whose isotropy representation on T_pM is irreducible. For a non-zero eigenvalue λ, the eigenspace $V_\lambda = \{f \in C^\infty(M) \mid \Delta f = \lambda f\}$ is finite-dimensional and $G = \mathrm{Isom}(M)$-invariant.

Choose an orthonormal basis $\{f_1, \ldots, f_n\}$ for V_λ, with respect to this inner product. We obtain a mapping $\tilde{x} : M \to \mathbb{R}^n$ by

$$\tilde{x}(p) = (f_1(p), \ldots, f_n(p)) \quad \text{for } p \in M. \tag{46}$$

The metric pulled back via $\tilde{x}$ is then:

$$\tilde{g} = \sum_i df_i \otimes df_i. \tag{47}$$

This pullback map is G-invariant, and so Schur's lemma forces $\tilde{g} = c^2 g$ with $c \neq 0$.

We can then rescale by c^{-1}, giving the map $x(p) = \frac{1}{c}\tilde{x}(p)$, which defines an isometric immersion of M into $\mathbb{R}^n$, satisfying $\Delta x = \lambda x$ with $\lambda \neq 0$. By Takahashi's prior theorem, this immersion is minimal into a sphere.

We conclude that every compact homogeneous Riemannian manifold with irreducible isotropy group admits a minimal isometric immersion into a Euclidean sphere. In particular, every irreducible compact symmetric space admits a minimal isometric immersion as well.

□

This result shows that highly symmetric spaces can be realised as minimal submanifolds of ambient Euclidean spaces, which offers a geometric lens from which one can view their structure.

Seeking an even broader principle, Song conjectured the following:

Conjecture 1 (Song)**.** *Every irreducible n-dimensional symmetric space admits an isometric immersion into the unit sphere $\mathcal{S}(\mathcal{H})$ of a seperable Hilbert space $\mathcal{H}$ as an n-dimensional minimal submanifold [Song, 2024].*

While open in full generality, Song's conjecture reduces in finite dimensions to Takahashi's theorem and related results of do Carmo–Wallach.

Combining these insights, with Jensen's classification, we arrive at the following:

Theorem 7 (Minimal realisation of locally homogeneous Einstein four-manifolds)**.** *Let (M^4, g) be an Einstein four-manifold that is locally homogeneous. Then:*

- *M is locally symmetric* (*Jensen, Theorem 5*).
- *If M is globally symmetric, and assuming Conjecture 1, it admits a minimal isometric immersion into a Hilbert sphere $\mathcal{S}(\mathcal{H})$.*
- *If M is compact, it admits a minimal immersion into a Euclidean sphere* (*Takahashi, Theorem 6*).

Proof. The first statement follows directly from Jensen's Theorem. The second is a consequence of Song's conjecture (assuming it holds), and the third is a corollary of Takahashi's result for compact symmetric spaces. □

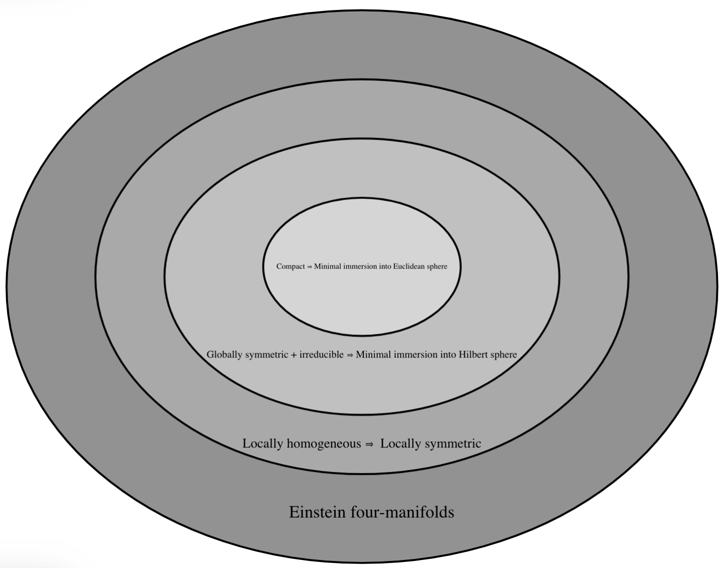

Figure 6: Implication diagram: local homogeneity $\Rightarrow$ local symmetry (Jensen); global symmetry $+$ Song $\Rightarrow$ Hilbert-sphere immersion; compactness $+$ Takahashi $\Rightarrow$ Euclidean-sphere immersion. (Diagram drawn with Draw.io)

8.3 A Concrete Minimal Immersion of an Einstein Manifold

A natural way to illustrate the ideas of this chapter is to present a specific example: the complex projective plane $\mathbb{CP}^2$, equipped with its normalised Fubini–Study metric, admits a minimal isometric immersion into the round sphere $S^7 \subset \mathbb{R}^8$. This example elegantly synthesises ideas from Riemannian, complex, and algebraic geometry.

8.3.1 The Geometry of $\mathbb{CP}^2$

Recall that $\mathbb{CP}^2$ is the space of complex lines through the origin in $\mathbb{C}^3$. Formally:

Definition 12 (Complex Projective Plane)**.** *The complex projective plane $\mathbb{CP}^2$ is the space of complex lines through the origin in $\mathbb{C}^3$:*

$$\mathbb{CP}^2 \;=\; \big(\mathbb{C}^3 \setminus \{0\}\big)\big/\big(z \sim \lambda z,\ \lambda \in \mathbb{C}^*\big) \tag{48}$$

[Harris, 1992].

This defines a smooth, compact, complex manifold of complex dimension 2 (or real dimension 4).

The distinguished Riemannian metric is the Fubini-Study metric, g_{FS}, which has the advantage of being Kähler and of constant holomorphic sectional curvature.

Definition 13 (Fubini-Study Metric)**.** *In the affine chart* $[1 : z_1 : z_2]$*, set:*

$$g_{i\bar{j}} \;=\; \partial_{z_i}\partial_{\bar{z}_j} \log\bigl(1 + |z_1|^2 + |z_2|^2\bigr), \qquad i, j = 1, 2. \tag{49}$$

This Hermitian metric is Kähler with constant holomorphic sectional curvature $+4$*.[Matsumoto, 2018].*

Note, we have introduced the notion of the Hermitian metric, so we include the following definition for completeness:

Definition 14 (Hermitian Metric)**.** *A Hermitian metric on a complex manifold* M *is a Riemannian metric* g *which satisfies:*

$$g(JX, JY) = g(X, Y), \quad \forall X, Y \in T_xM,\ \forall x \in M, \tag{50}$$

where J *denotes the amost complex structure on* M*.*

8.3.2 The Veronese Embedding

In order to embed $\mathbb{CP}^2$ as a submanifold of a sphere, the Veronese embedding is introduced, which is a classical method to realise projective spaces as algebraic subvarieties of higher-dimensional projective space.

Definition 15 (Veronese Embedding)**.** *The* d*-uple Veronese embedding is the morphism:*

$$\nu_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N, \qquad [x_0 : \ldots : x_n] \mapsto \bigl[\textit{all monomials of degree } d\bigr]. \tag{51}$$

This map realises $\mathbb{P}^n$ *as a smooth projective variety of degree* d^n *in* $\mathbb{P}^N$*, where* $N = (n + d) - 1$ *[Harris, 1992, Shafarevich and Reid, 1994].*

The most fundamental example is taking $n = 2, d = 2$, which gives the classical Veronese surface:

$$\nu_2 : \mathbb{CP}^2 \longrightarrow \mathbb{CP}^5, \quad [x : y : z] \longmapsto [x^2 : xy : xz : y^2 : yz : z^2]. \tag{52}$$

We would like to explore this in more detail.

8.3.3 Lifting to the Sphere and Minimality

Choose homogeneous coordinates with unit length:

$$\tilde{\nu}_2([x : y : z]) = \frac{1}{\|v\|}\bigl(x^2,\, xy,\, xz,\, y^2,\, yz,\, z^2\bigr) \in \mathbb{C}^6, \qquad \|v\|^2 = |x|^4 + |x|^2|y|^2 + \cdots + |z|^4. \tag{53}$$

This defines a map:

$$\tilde{\nu}_2 : \mathbb{CP}^2 \to \mathbb{S}^{11} \subset \mathbb{C}^6, \tag{54}$$

which is a horizontal lift for the Hopf fibration:

$$\nu_2 = \pi \circ \tilde{\nu}_2. \tag{55}$$

Here, $\pi : \mathbb{S}^{11} \to \mathbb{CP}^5$ denotes the generalised Hopf fibration, with $\nu_2 = \pi \circ \tilde{\nu}_2$.

Definition 16 (Hopf Fibration)**.** *The Hopf fibration is the smooth surjective map*

$$\pi : \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1} \to \mathbb{CP}^n, \quad z \mapsto [z],$$

which sends each point $z \in \mathbb{S}^{2n+1}$ *to its complex line through the origin. The fibers are circles* $\mathbb{S}^1$*, and this realizes* $\mathbb{S}^{2n+1}$ *as a principal* $\mathbb{S}^1$*-bundle over* $\mathbb{CP}^n$ *[Frankel, 2011, Kobayashi and Nomizu, 1969].*

This structure underlies the Veronese embedding's lift: by normalising homogeneous coordinates, one maps $\mathbb{CP}^2$ into $\mathbb{S}^{11} \subset \mathbb{C}^6$, and the Hopf fibration ensures that this lift descends correctly back to projective space.

8.3.4 Minimal Immersion into S^7

Because the image of $\tilde{\nu}_2$ is orthogonal to the circle fibres, standard submersion theory shows that $\tilde{\nu}_2$ is minimal in S^{11} if and only if ν_2 is minimal in $\mathbb{CP}^5$.

Finally, identify $\mathbb{C}^6 \cong \mathbb{R}^{12}$ and observe that the real and imaginary parts of the six complex coordinates span an 8-dimensional real linear subspace $\mathbb{R}^8 \subset \mathbb{R}^{12}$. The image of $\tilde{\nu}_2$ lies entirely inside the intersection $S^{11} \cap \mathbb{R}^8 = S^7$, and the induced metric equals g_{FS}. Hence we arrive at:

Proposition 1. $\left(\mathbb{CP}^2,\, g_{\mathrm{FS}}\right)$ *admits a minimal isometric immersion into the round sphere* $S^7 \subset \mathbb{R}^8$.

Carmo–Wallach showed that this immersion is characterised by the first non-trivial eigenspace of the Laplacian on $(\mathbb{CP}^2, g_{\mathrm{FS}})$ [do Carmo and Wallach, 1971].

This example concretely demonstrates the key theme of this exposition: that certain Einstein manifolds, particularly those with high symmetry, can indeed be realised as minimal submanifolds of ambient spaces. It gives not only a conceptual link between curvature variational theories, but also a precise geometric construction bridging the two domains.

9 Conclusion and Outlook for Future Research

This exposition set out to understand the geometric parallels between minimal surfaces and Einstein four-manifolds. Although these objects arise in different dimensions and distinct corners of geometry, they stand on common ground.

Through a shared variational structure, second variation theory, stability analysis, monotonicity properties, and compactness results, both minimal surfaces and Einstein manifolds reveal a common analytical and topological theme. Both admit natural decompositions, into sheeted and non-sheeted, or thick and thin regions, which isolate geometric regularity from degeneration, and allow powerful theorems to emerge to aid in analysis.

These parallels are not merely structural: under symmetry, an Einstein manifold can be isometrically immersed as a minimal submanifold of a higher dimension. The explicit immersion of $\mathbb{CP}^2$ into the round sphere demonstrated that in special cases, the two theories fuse. More broadly, this project offered a perspective on how ideas in geometric analysis can echo across different settings.

There are, of course, many questions left open. Can similar analogies be found in other elliptic systems? Might these ideas extend to gauge theory? Could further study of symmetric spaces lead to new classification results for Einstein manifolds?

In closing, this has been both a journey of analysis and philosophy. It shows that even in the abstract world of differential geometry, there are moments where distinct objects from different spaces can speak the same language. Einstein four-manifolds and minimal surfaces, drawn from different dimensions but sharing crucial parallels, is a reminder that order pervails in mathematics.

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