DPhil Summer Work

Curvature Determines Holonomy: Restricted Holonomy and the Ambrose-Singer Theorem

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OVERVIEW

WHERE DID WE LEAVE OUR STUDIES LAST TIME?

Warm-up: Suppose that (M, ∇) is a manifold with a connection (on a vector or principal bundle).

- What do we mean by the restricted holonomy group at a point $p \in M$?
- What is the motivation for its introduction?

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The motivation around this is that if the manifold is not simply-connected, loops can differ in their homotopy class. Full holonomy accounts for all of these, but restricted holonomy ignores this and focuses only on the local structure. A good philosophy is then: $\mathrm{Hol}^0(g)$ teaches us about the manifold structure when the dimension of $Hol^0(g)$ is quite small.

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BUILDING UP TO THE AMBROSE-SINGER THEOREM

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To do so, we need to lay the groundworks, exploring curvature and its relation to the connection.

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Imagine then that you are considering a curved surface, equipped with an arrow. You draw a loop on the surface, and place the arrow at a starting point on the loop. You transport the arrow around the loop until it returns to the starting point.

- If the surface is flat, what happens when you return to the starting point?
- If the surface is curved, what happens when you return to the starting point?

CURVATURE AND HOLONOMY

A good suggestion would be:

- If the surface is flat, like a plane, then the arrow returns to its original orientation. I.E. The restricted holonomy group is trivial!
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The link with holonomy is then immediate: that rotation in the latter case is an element of the holonomy group, and the size and direction of that rotation (when the loop is infinitesimal) are determined by the curvature.

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Parallel transport a vector v around $\gamma_{X,Y}$. The change in v is given approximately by:

$$v \mapsto v - R(X, Y)v \cdot (\text{Area}) \tag{1}$$

This tells us that the curvature tensor acts like an infinitesimal generator of the holonomy!

LINKING TO LIE GROUPS

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Translation: *Small loops generate small elements of holonomy, and those come from curvature. Over time, transporting many such loops in various directions builds up the holonomy group, and this is what the Ambrose-Singer Theorem formalises.*

MOTIVATION

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2. Curvature encodes infinitesimal holonomy
3. Determining the holonomy Lie algebra

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1. How does parallel transport behave around small loops?
2. Curvature encodes infinitesimal holonomy
3. Determining the holonomy Lie algebra

The Ambrose-Singer Theorem gives us a remarkable answer: The curvature tensor generates the holonomy Lie algebra. Precisely, we can build up every infinitesimal holonomy transformation by looking at the curvature endomorphisms, parallel transported back to the base point.

THE AMBROSE-SINGER THEOREM, 1953 (VECTOR BUNDLES)

Let $E \to M$ be a vector bundle with connection ∇, and fix $p \in M$. Then the Lie algebra of the restricted holonomy group at p, denoted $\mathfrak{hol}_p$, is the smallest Lie subalgebra of $\mathrm{End}(E_p)$, containing all endomorphisms of the form:

$$\tau_\gamma^{-1} \circ R_x(Y, Z) \circ \tau_\gamma, \tag{3}$$

where $x \in M$, γ is a smooth path from p to x, and $Y, Z \in T_x M$

THE AMBROSE-SINGER THEOREM, 1953 (PRINCIPAL BUNDLES)

A more conceptual version of the theorem can also be presented, using the idea of principal bundles.
Let M be a manifold, P be a principal bundle over M with fibre G, and D be a connection on P. Fix $p \in P$, and define $Q = \{q \in P : p \sim q\}$.
Then, $\mathfrak{hol}_p(P, D)$ is the vector subspace of the Lie algebra $\mathfrak{g}$ of G spanned by the elements of the form $\pi^*(R(P, D) \cdot v \wedge w)_q$ for all $q \in Q$ and $v, w \in C^\infty(TM)$, where π maps $P \times \mathfrak{g} \mapsto \mathrm{ad}(P)$.

REMARKS ON THE AMBROSE-SINGER THEOREM

We have two perspectives on Ambrose-Singer:

- Vector bundle: Curvature as endomorphism, transported to basepoint
- Principal bundle: Curvature as vertical bracket of horizontal lifts. Each gives rise to the same Lie algebra of the restricted holonomy group.

SKETCH OF PROOF FOR THE AMBROSE-SINGER THEOREM

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- Take a loop based at p, which is homotopic to zero.
- Fill up the loop with a smooth surface.
- Use a typical finite decomposition of the lasso, applying the 'Lasso Lemma', by a double integral on the surface.
- This yields the parallel translation along the loop as a double integral of the desired form.