On the Parallels Between Minimal Surfaces and Einstein Four-Manifolds

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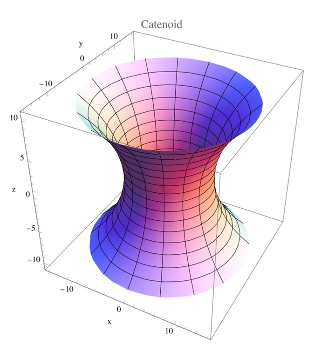
Minimal Surfaces

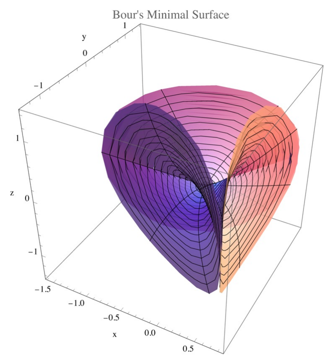
Definition

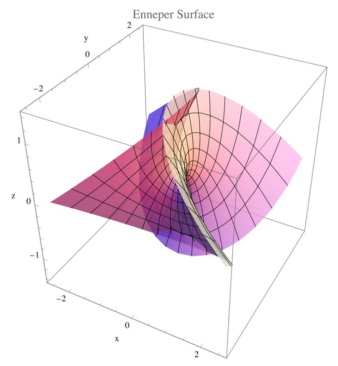
A surface in $\mathbb{R}^3$ is minimal if and only if its mean curvature vanishes at every point. That is:

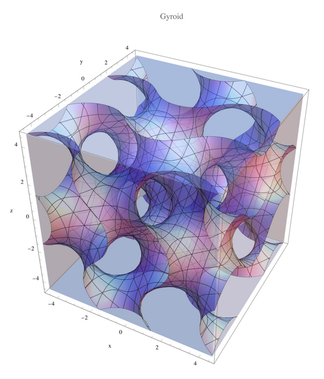
$$H = \frac{1}{2}(k_1 + k_2) = 0 \iff k_1 = -k_2,$$

where k_1 and k_2 are principal curvatures of the surface.









Variational Approach

Variationally, minimal surfaces are critical points of the area functional:

$$\mathcal{A} = \int_M \mathrm{d}A,$$

Further, minimal surfaces are fixed points of the Mean Curvature Flow (MCF)

$$\frac{\partial X}{\partial t} = -H\nu,$$

Second Variations

The second-variation of the area functional:

$$\delta^2 \mathcal{A}(V,V) = \int_M \langle \mathcal{J}V, V \rangle \, \mathrm{d}\mu.$$

where J is the Jacobi operator:

$$\mathcal{J} = \Delta^{\perp} + |A|^2 + \mathrm{Ric}_N(\nu,\nu),$$

Monotonicity and Local Rigidity

For any point $p \in \Sigma$ and sufficiently small $r > 0$, the function:

$$r \mapsto \frac{\mathrm{Area}(\Sigma \cap B(p,r))}{\pi r^2},$$

is monotone non-decreasing up to curvature correction terms when the ambient space N is not flat

Epsilon Regularity and Energy Thresholds

Choi-Schoen: There exist constants $\varepsilon > 0$ and $C > 0$ such that the following holds. Let Σ be an embedded minimal surface in a Riemannian 3-manifold N and let $p \in \Sigma$. If $r > 0$ is sufficiently small and:

$$\int_{\Sigma \cap B(p,r)} |A|^2 < \varepsilon,$$

then:

$$\sup_{\Sigma \cap B(p,r/2)} |A|^2 \leq \frac{C}{r^2},$$

Compactness and Convergence

Sharp-Chodosh-Ketover-Maximo: Let $\{\Sigma_i\}$ be a sequence of closed, embedded minimal surfaces in a Riemannian 3-manifold (N, g), such that:

$$\mathrm{Area}(\Sigma_i) \leq C, \qquad \mathrm{index}(\Sigma_i) \leq C.$$

Then, after passing to a subsequence $\Sigma_i \to \Sigma_\infty \subset N$ smoothly on compact subsets. Furthermore, the genus of Σ_i is uniformly bounded, and the limit Σ_∞ is a smooth, embedded minimal surface

(For references, please see the paper on my website, as there are too many to list here)

Einstein Four-Manifolds

Definition

A Riemannian manifold (M, g) is Einstein if there exists a constant $\lambda \in \mathbb{R}$ such that:

$$\mathrm{Ric}_g = \lambda g,$$

where Ric is the Ricci curvature tensor.

d = 2 A Riemannian manifold is Einstein if and if it has constant Gaussian curvature

d = 3 A Riemannian manifold is Einstein if and only if it has constant sectional curvature

d = 4 Best tool is elimination. The chief tool is the Hitchin-Thorpe inequality:

$$|\tau(M)| \leq \frac{2}{3}\chi(M),$$

d > 5 Usually admit a negative Einstein metric. Existence in positive case hinges on Myers theorem and the Yamabe problem.

Variational Approach

Einstein four-metrics are critical points of the Einstein-Hilbert functional:

$$\mathcal{E}(g) = \frac{\int_M R_g \mathrm{dvol}_g}{(\mathrm{Vol}(M,g))^{1/2}},$$

Einstein metrics arise as fixed points of the Ricci flow

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g(t)),$$

Second Variations

Second-variation of the Einstein-Hilbert functional, restricted to transverse-traceless tensors, is governed by the Lichnerowicz Laplacian:

$$\Delta_L h = \nabla^* \nabla h + 2\mathring{R}h,$$

Monotonicity and Local Rigidity

For any $p \in M$ and sufficiently small $r > 0$, the normalised volume function:

$$r \mapsto \frac{\mathrm{Vol}(B(p,r))}{r^4},$$

is almost monotone non-increasing as $r \to 0$, with equality if and only if the metric is locally flat at p

Epsilon-Regularity and Energy Thresholds

Anderson, Cheeger-Tian, Nakajima, Gao: There exist constants $\varepsilon > 0$ and $C > 0$ such that the following holds. Let (M^4, g) be an Einstein manifold, and let $p \in M$. If $r > 0$ is sufficiently small and $\int_{B(p,r)} |Rm|^2 < \varepsilon,$ then:

$$\sup_{B(p,r/2)} |Rm| \leq \frac{C}{r^2},$$

Compactness and Convergence

Anderson-Bando-Nakajima-Gao: Let $\{(M_i, g_i)\}$ be a sequence of Einstein four-manifolds satisfying:

$$\chi(M_i) \leq C, \quad \mathrm{Vol}(M_i, g_i) \geq C^{-1}, \quad \mathrm{diam}(M_i, g_i) \leq C.$$

Then after passing to a subsequence, $(M_i, g_i) \to (X, g_\infty)$ in the Gromov-Hausdorff, where X is a smooth Einstein orbifold. Furthermore, the number of diffeomorphism types of M_i is finite.

Embedding Einstein Four-Manifolds as Minimal Submanifolds

Why?

The parallels proposed span the fields of topology, differential geometry, and geometric analysis. It is thus natural to consider the conceptual link between the objects. A fruitful question to ask is whether, under certain conditions, an Einstein four-manifold may be isometrically immersed as a minimal submanifold in a higher-dimensional ambient space.

Jensen's Theorem

Recall that a Riemannian manifold (M, g) is locally symmetric if its Riemann curvature tensor is parallel:

$$\nabla R = 0$$

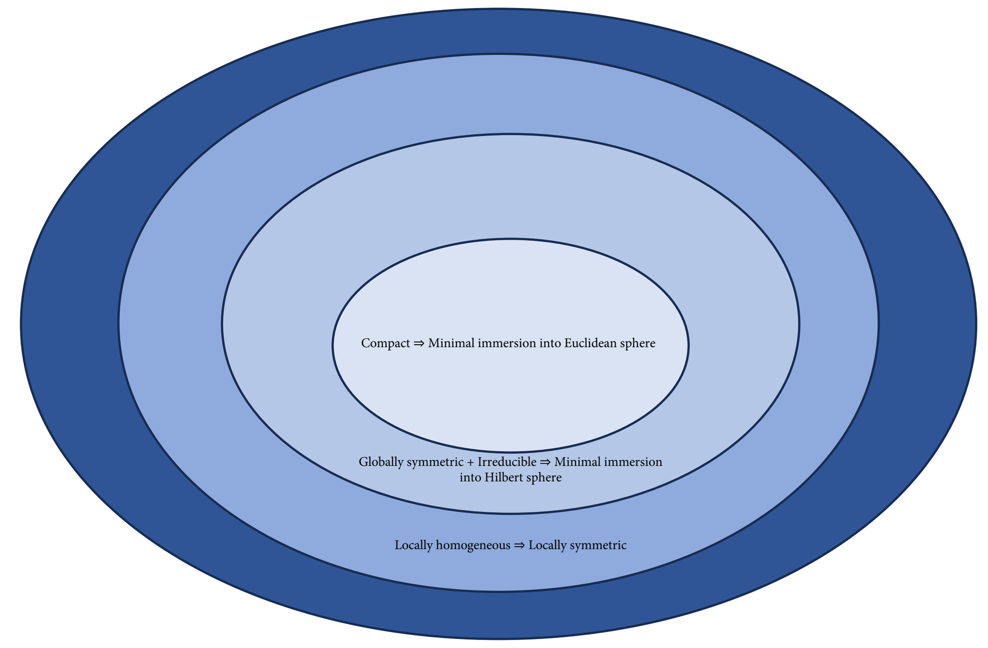
Every locally homogeneous Einstein four-manifold is locally symmetric (Jensen, 1969).

Takahashi's Theorem

Every irreducible compact symmetric space admits a minimal isometric immersion into a Euclidean space (Takahashi, 1966).

Song's Conjecture

Every irreducible n-dimensional symmetric space admits an isometric immersion into the unit sphere $\mathcal{S}(\mathcal{H})$ of a separable Hilbert space $\mathcal{H}$ as an n-dimensional minimal submanifold (this reduces in finite dimensions to Takahashi's theorem and the related results of do Carmo-Wallach)



Concrete Example

Consider the complex projective plane, $\mathbb{C}P^2$, equipped with its normalised Fubini-Study metric:

$$g_{i\bar{j}} = \partial_{z_i}\partial_{\bar{z}_j} \log\left(1 + |z_1|^2 + |z_2|^2\right), \qquad i,j = 1,2.$$

This admits a minimal isometric immersion into the round sphere $S^7 \subset \mathbb{R}^8$. We realise this by defining a map:

$$\tilde{\nu}_2 : \mathbb{CP}^2 \to \mathbb{S}^{11} \subset \mathbb{C}^6,$$

which is a horizontal lift for the Hopf fibration,

$$\nu_2 = \pi \circ \tilde{\nu}_2.$$

Because the image of this map is orthogonal to the circle fibres, standard submersion theory shows that it is minimal in S^{11} if and only if the Hopf fibration is minimal in $\mathbb{C}P^2$. Then, identifying $\mathbb{C}^6 \cong \mathbb{R}^{12}$, we cam realise the real and imaginary parts of the six complex coordinates span an 8-dimensional real linear subspace. The image of the map then lies entirely in the intersection $S^{11} \cap \mathbb{R}^8 = S^7$, so we see that $(\mathbb{C}P^2, g_{FS})$ admits a minimal immersion into the round sphere.