DPhil Summer Work

# Introduction to Holonomy: Vector Bundles, Principal Bundles, Connections, Holonomy Definition

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# OVERVIEW

# WHAT INTUITION SHOULD WE CARRY FORWARD?

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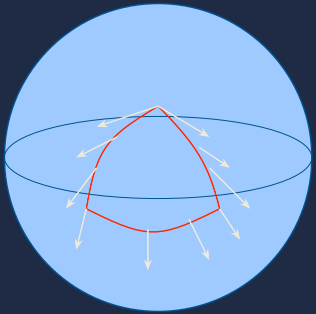
This is of use to us because it can help us detect details on the geometry, for example whether there is a hole or a region of high curvature.



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This change in the pointing of your thumb is a demonstration of holonomy!

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The net horizontal shift is holonomy!

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# VECTOR BUNDLES AND PRINCIPAL BUNDLES

Let's briefly refresh our mind of the concepts that we need, in order to understand holonomy.

We will be defining (in this presentation) holonomy on vector bundles and principal bundles, which is the most general scenario that one can consider. We will then reward ourselves by considering the more familiar example of Riemannian geometry, which we know from experience comes with pleasent definitions of connections and curvature.

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*(Recall that our reasoning for introducing this rather abstract concept was to conceptualise the tangent space and cotangent space as a manifold)*

Formally, a real vector bundle of rank $k$ over a manifold $M$ consists of a manifold $E$, with a surjective continuous map $\pi : E \to M$, satisfying:

1. For each $p \in M$, the fibre $E_p = \pi^{-1}(p)$ over $p$ is endowed with the structure of a $k$-dimensional real vector space.
2. For each $p \in M$ there exists a neighbourhood $U$ of $p$ in $M$ and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, which we call a local trivialisation of $E$ over $U$.

Further, we require that the local trivialisation satisfies:

- For a projection $\pi_U : U \times \mathbb{R}^k \to U$, $\pi_U \circ \Phi = \pi$
- For each $q \in U$, the restriction $\Phi$ to $E_q$ is a vector space isomorphism from $E_q$ to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$

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To be somewhat specific, in the concept of the vector bundle, $M$ and $E$ were general manifolds, but the fibres $\pi^{-1}(x)$ were required to be vector spaces. We'd like to generalise this, so that the fibre $\pi^{-1}(x)$ can be any general manifold.

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As such, we consider that the fibre is a Lie group. This is of course still more general than the vector bundle, but specific enough that it is useful for descriptions. And of course, Lie groups are much studied, and we have a decent idea of the underlying theorems and such.

## PRINCIPAL BUNDLES

Let $G$ be a Lie group, and let $P$ and $M$ be smooth manifolds. A principal $G$-bundle is a quadruple $(P, M, \pi, G)$, where:

1. $\pi : P \to M$ is a smooth surjective submersion
2. $G$ Acts freely and smoothly on the right of $P$:
$$R_g : P \to P, \quad p \mapsto p \cdot g. \tag{1}$$
That is, every group element moves a point in $P$ without fixing it.
3. The action is fibre-preserving, and the fibres of $\pi$ are the orbits of action:
$$\pi(p \cdot g) = \pi(p), \quad \forall p \in P, g \in G. \tag{2}$$
4. It is locally trivial, so for every point $x \in M$, there exists an open neighbourhood $U \subset M$, such that:
$$\pi^{-1}(U) \cong U \times G. \tag{3}$$

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The issue we should immediately see is that the vectors live in different spaces, and thus there is no natural way to say that a vector over $p$ is the same as one over $q$.

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The connection provides us with such a tool. It gives us a way to differentiate sections, taking into account the geometry of each bundle.

Let $E \to M$ be a smooth vector bundle. Let $\Gamma(E)$ be the space of smooth sections of $E \to M$. A connection on $E$ is a map $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$, such that:

$$\nabla(fs) = df \otimes s + f\nabla s, \quad \forall f \in C^\infty(M), s \in \Gamma(E), \tag{4}$$

and further:

$$\nabla s_1 + s_2 = \nabla s_1 + \nabla s_2 \tag{5}$$

That is, a connection on a vector bundle is a derivation in the section argument and a $C^\infty(M)$-linear map in the vector field. In that way, it is like a covariant derivative, telling us how the section is changing relative to the bundle.

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On occassion, it will be of use to us to consider the connection locally. In the local frame $\{e_i\}$ for the bundle $E$, you can write any section $s = s^i e_i$. Then a connection acts as:

$$\nabla_X s = X(s^i)e_i + s^i \nabla_X e_i \tag{6}$$

## CONNECTION ON A PRINCIPAL BUNDLE

Take a principal bundle $\pi : P \to M$, with structure group $G$. Over each point $x \in M$, the fibre $\pi^{-1}(x)$ is a copy of $G$, and the bundle comes equipped with a right $G$-action that's free and transitive on each fibre.

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Our question this time arises from the following: If you are walking along the manifold $M$ carrying elements of $G$, how do you move in a horizontal direction, when everything you know is vertical along fibres?

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Further, how do you know when you are going in a direction which is not just twisting around the group, but is really exploring the geometry of the base manifold?

Again, the concept of a connection answers this! Although it does something different this time: It gives you a way to split the tangent space of the total space $P$ into:

$$T_pP = H_pP \oplus V_pP, \tag{7}$$

where:

- $V_pP$ Is the vertical subspace, which are the directions that you move along the fibre
- $H_pP$ Is the horizontal subspace which is defined by the connection, which tells us how to stay level with respect to the base space $M$

Let $\pi : P \to M$ be a smooth principal $G$-bundle, with right action $R_g : P \to P$ for each $g \in G$. The vertical subspace at a point $p \in P$ is:

$$V_pP := \mathrm{ker}(d\pi_p) \subset T_pP. \tag{8}$$

These are the directions which are tangent to the fibre through $p$, and they correspond to the Lie algebra $\mathfrak{g}$ of $G$. The connection gives a horizontal complement, $H_pP$, which varies smoothly in $p$, such that:

$$T_pP = H_pP \oplus V_pP, \quad \forall p \in P. \tag{9}$$

The assigment $p \mapsto H_pP$ is $G$-equivalent, meaning that the horizontal spaces transform nicely under the right-action:

$$(dR_g)_p(H_pP) = H_{p\cdot g}P \tag{10}$$

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Parallel transport!

We first ask, how do we know if a section is parallel?

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# PARALLEL TRANSPORT ON A VECTOR BUNDLE

For the construct that follows, let $M$ be a smooth manifold, and let $E \to M$ be a vector bundle over $M$. Denote the connection on $E$ with $\nabla^E$.

Let $\gamma : [0,1] \to M$ be a smooth curve in $M$. We may pullback $\gamma$, giving us $\gamma^*(E)$, which takes us from $E$ to $[0,1]$.
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Let $s$ be a smooth section of $\gamma^*(E)$ over $[0, 1]$, so that $s(t) \in E_{\gamma(t)}$ for each $t \in [0, 1]$.

We get a connection on $\gamma^*(E)$ for 'free', as $\nabla^E$ pulls back under $\gamma$ to give a connection on $\gamma^*(E)$ over $[0, 1]$.

Using this, our definition for parallel transport takes the following form: $s$ Is parallel if its derivative under the pulled-back connection is 0. That is, if $\nabla^E_{\dot{\gamma}(t)} s(t) = 0, \quad \forall t \in [0,1]$.

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Let us remark that the parallel transport equation is a first-order ODE in $s(t)$. This means that for each initial value $e \in E_{\gamma(0)}$, there exists a unique, smooth solution $s$ with $s(0) = e$, VIA the Picard-Lindelöf Theorem.

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Let $M$ be a smooth manifold, $E$ be a vector bundle over $M$, and $\nabla^E$ be a connection on $E$. Suppose that $\gamma : [0,1] \to M$ is smooth, with conditions that $\gamma(0) = x, \gamma(1) = y$, for $x, y \in M$. Then for each $e \in E_x$, there exists a unique smooth section $s$ of $\gamma^*(E)$, satisfying $\nabla^E_{\dot{\gamma}(t)} s(t) = 0$, for $t \in [0,1]$, and $s(0) = e$. Define $P_\gamma(e) = s(1)$. The parallel transport map is the well-defined linear map $P_\gamma : E_x \to E_y$.

# HORIZONTAL LIFTS ON PRINCIPAL BUNDLES

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Let us set things up like so: Let $M$ be a manifold, $P$ be a principal bundle over $M$ with structure group $G$, and $D$ a connection on $P$.
Let $\gamma : [0,1] \to P$ be a smooth curve in $P$. Then $\dot{\gamma} \in T_{\gamma(t)}P$ is tangent to $\gamma([0,1])$ for each $t \in [0,1]$.
$\gamma$ Is a horizontal curve if its tangent vectors are horizontal ($\dot{\gamma}(t) \in D_{\gamma(t)}$ for each $t \in [0,1]$.

The definition extends to the scenerio of $\gamma : [0,1] \to P$ being piecewise-smooth, with the classification then being that $\gamma$ is horizontal if $\dot{\gamma}(t) \in D_{\gamma(t)}$ for $t$ in the open, dense subset of $[0,1]$, for $\dot{\gamma}$ well-defined.

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Indeed, if $\gamma : [0,1] \to M$ is piecewise-smooth, with $\gamma(0) = m$, and $p \in P$, with $\pi(p) = m$, then there exists a unique horizontal, piecewise-smooth map $\gamma' : [0,1] \to P$, such that $\gamma'(0) = 0, \pi \circ \gamma'$ is equal to $\gamma$.
This follows from the Picard-Lindelof Theorem for ODEs, in a way analagous to the use of this theorem for parallel sections of $\gamma^*(E)$, as in the case of vector bundles.
We define $\gamma'$ as a horizontal lift of $\gamma$.

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The holonomy group $\mathrm{Hol}_x(\nabla^E)$ of $\nabla^E$ based at $x$ is defined as:

$$\mathrm{Hol}_x(\nabla^E) = \{P_\gamma : \gamma \text{ Is a loop based at } x\} \subset \mathrm{GL}(E_x) \quad (11)$$

We note a particular type of holonomy group, called the restricted holonomy group. The restricted holonomy group based at $x$ is the subgroup $\mathrm{Hol}_x^0(\nabla)$, where we consider only the loops $\gamma$ that are contractible.

## HOLONOMY GROUP FOR PRINCIPAL BUNDLES

We proceed in a somewhat parallel manner. Let $G$ be a Lie group, and $P$ be a principal $G$-bundle over a smooth manifold $M$, and let $D$ be a connection on $P$.

For $p, q \in P$, we write $p \sim q$ if there exists a piecewise-smooth horizontal curve in $P$ which joins $p$ and $q$, which defines an equivalence relation.

Fix $p \in P$, and define the holonomy groups of $(P, D)$ based at $p$ to be:

$$\mathrm{Hol}_p(P, D) = \{g \in G : p \sim g \cdot p\}. \tag{12}$$

We can define the restricted holonomy as before, denoted as $\mathrm{Hol}_p^0(P, D)$, for which we consider its elements the horizontal lifts of contractible loops $\gamma$.

# HOLONOMY GROUP FOR PRINCIPAL BUNDLES

Does our idea of connectedness from the vector bundle scenario follow through here?

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Yes! If $M$ and $P$ are connected, then the holonomy group depends on the basepoint $p$ only up to conjugation in $G$. I.E. If we take $q$ to be any other basepoint for the holonomy, then there exists a unique $g \in G$ such that $q \sim p \cdot g$. With such value of $g$:

$$\mathrm{Hol}_q(\nabla) = g^{-1}\,\mathrm{Hol}_p(\nabla)g, \tag{13}$$

and in particular:

$$\mathrm{Hol}_{p \cdot g}(\nabla) = g^{-1}\,\mathrm{Hol}_p(\nabla)g. \tag{14}$$

## NEXT TIME...

- Fundamental properties of holonomy, with proofs
- Ambrose-Singer Theorem
- Berger's classification.