§4. RINGED SPACES:

We are now almost in the position to define SCHEMES, starting from affine schemes which are building blocks

Taking a step back, we thus far had a RING and associated SPEC(R) to it, along with a SHEAF OF RINGS, $\mathcal{O}$

Let's consider the idea of a category, with $(X, \mathcal{O}_X)$ as the objects

To do so, we note that $(\mathrm{Spec}(R), \mathcal{O})$ is a RINGED SPACE, where a ringed space is, in general, a pair $(X, \mathcal{O}_X)$, with X a topological space and $\mathcal{O}_X$ a sheaf of rings

We may construct morphisms of our $(\operatorname{Spec} R, \mathcal{O})$, first recalling this definition in general.

DEFINITION: A MORPHISM OF RINGED SPACES from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ of continuous maps:

- $f : X \longrightarrow Y$ continuous
- $f^\# : \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$ Map of sheaves

REMARK: $(\operatorname{Spec}(R), \mathcal{O})$ IS such that every stalk is a local ring

(LOCALLY RINGED means, for $(X, \mathcal{O}_X)$, each $p \in X$ is such that $\mathcal{O}_{X,p}$ is a local ring)

Any morphisms we define on local ringed spaces, we would like to respect our property that each stalk is a local ring - so cautious definitiom needed!

DEFINITION: Let A, B be local rings, where they have maximal ideal m_A and m_B.
A homomorphism $\varphi: A \to B$ is a LOCAL HOMOMORPHISM if $\varphi^{-1}(m_B) = m_A$.

This now puts us in the a position to define schemes!

§5. SCHEMES:

§5.1 DEFINITIONS:

DEFINITION: An AFFINE SCHEME is a locally ringed space isomorphic to

$(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$, for a ring, A.

The definition for a scheme naturally follows!

DEFINITION: A SCHEME is a locally ringed space which is locally isomorphic to an affine scheme.

I.E. Every point has an open neighbourhood U, such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

Two bricks which schemes are built from:

Sheaf of rings, $\mathcal{O}_X$, restricts to structure sheaves, $\mathcal{O}_{\mathrm{Spec}(A)}$

Top. space, X, with covering consisting of open sets of form $\mathrm{Spec}(A_i)$

DEFINITION: A MORPHISM of schemes is a morphism of the locally ringed spaces

§ 5.2 FIRST EXAMPLES:

- Example 1: Spec (k)

  If k is a field, then Spec (k) is an affine scheme

  Its topological space consists of one point

- Example 2: Affine line

  k Be a field

  Affine line over k, $\mathbb{A}^1_k$, is Spec $(k[x])$

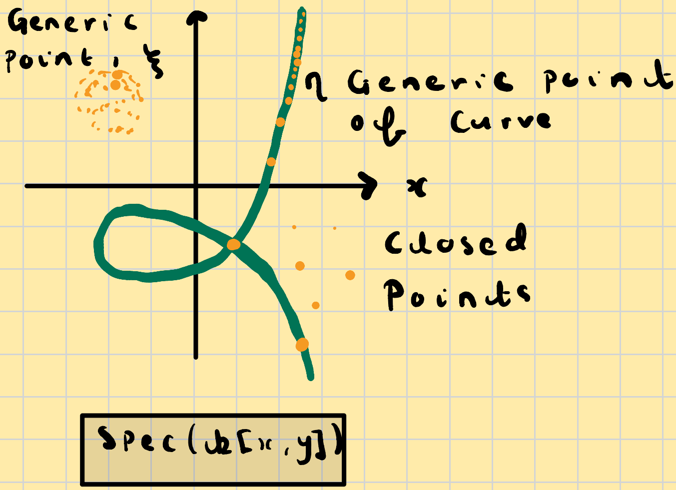
  Point is ξ, which corresponds to zero ideal - closure is the whole space (ξ Is generic point)

  Other points correspond to maximal ideals in $k[x]$ - are closed

• Example 3 : Affine plane

k := Algebraically closed field

$\mathbb{A}^2_k = \operatorname{Spec}(k[x,y])$



As shown above, for each irreducible polynomial $f(x,y)$, there is a point η whose closure consists of η and all closed points (a,b), for which $f(a,b)=0$