LECTURE FOUR (STALKS, MORPHISMS, AND SHEAFIFIC-ATION)

§3.3. STALKS:

IDEA: Take presheaf, $\mathcal{F}$, of abelian groups on a topological space, X $\longrightarrow$ Associate to each point $x \in X$ an abelian group, $\mathcal{F}_x$, the **STALK**

We take a small neighbourhood around $x \in X$, and the stalk gives us information of the section in this neighbourhood.

HOW?: Look at sections of open sets which contain $x \in X$ that agree on intersection - requires idea of **DIRECT LIMIT**

The **STALK**, $\mathcal{F}_p$ of $\mathcal{F}$ at $x \in X$, is the **DIRECT LIMIT** of $\mathcal{F}(U)$ for all $U \subseteq X$ containing x.

$$\mathcal{F}_p := \lim_{x \in U} \mathcal{F}(U) := \{(U, f \in \mathcal{F}(U))\} / \sim$$

$\sim :=$ $(U, f \in \mathcal{F}(U)) \sim (V, g \in \mathcal{F}(V))$ If there exists an open set $W \subseteq U \cap V$ which contains x, such that:

$f|_W = g|_W$

REMARK: $\sim$ is reflexive, symmetric, and transitive (if $(s,U)\sim(s,U')$ and $(s',U')\sim(s'',U'')$, one can find open neighbourhoods $V \subset U \cap U'$ and $V' \subset U' \cap U''$ of x, over which s and s', and s' and s'', coincide. s and s'' coincide on $V' \cap V$

DEFINITION: The **STALK**, $\mathcal{F}_x$, at $x \in X$ is the set of equivalence classes:

$$\mathcal{F}_x = \coprod_{x \in U} \mathcal{F}(U) / \sim$$

An element $s_x \in \mathcal{F}_x$ can be represented by a section $s \in \mathcal{F}(U)$ for $x \in U$, and s_x is the **GERM** of the section s at x.

In other words, the germ of s at x is the equivalence class, where for any neighbourhood U of x, there is a natural map $\mathcal{F}(U) \to \mathcal{F}_x$, which sends the section s to the equivalence class, where (s,U) belongs

GUIDING PRINCIPLE:

- The germ, s_x, of a section, s, vanishes $\Leftrightarrow$ The section vanishes on some neighbourhood of point x
- All elements of $\mathcal{F}_x$ are germs
- $\mathcal{F}$ is the zero sheaf $\Leftrightarrow$ All stalks zero

§3.4. MORPHISMS BETWEEN SHEAVES AND (PRE) SHEAVES:

DEFINITION: Let X be a topological space, and let F and G be (pre) sheaves on X. A **MORPHISM**, $\varphi : F \to G$, consists of morphisms:

$$\varphi(U) : F(U) \to G(U)$$

For each open set, such that, if $V \subseteq U$, then:

$$\begin{tikzcd}
F(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_{UV}"'] & G(U) \arrow[d, "\rho_{UV}"] \\
F(V) \arrow[r, "\varphi(V)"'] & G(V)
\end{tikzcd}$$

Commutes.

REMARK: Isomorphic if there exists an inverse to φ, such that:

$$\varphi \circ \varphi^{-1} = \mathbb{1}_G, \quad \varphi^{-1} \circ \varphi = \mathbb{1}_F$$

REMARK: (Pre) sheaves of abelian groups on X, with their morphisms forms a CATEGORY.

Also, sheaves of abelian groups forms a category

§3.5. PUSHFORWARD OF A SHEAF:

Let X be a topological space, $\mathcal{F}$ a sheaf on X, and $F: X \to Y$ a continuous map. One may define a sheaf on Y, $f_* F$ on Y by:

$$(f_* F)(U) = F(f^{-1} U)$$

The restriction maps come from F

One calls $f^* F$ a PUSHFORWARD

EXAMPLE: $p \in Y$ Fixed point

$f: X \to Y$ Constant map. I.E. $f(x) = p \quad \forall x \in Y$

Let F be sheaf on X

For $U \subseteq Y$, one has:

$$f_* F(U) = \begin{cases} F(x), & \text{If } p \in U \\ 0, & \text{If } p \notin U \end{cases}$$

If one is interested in the scenerio where G is a sheaf on Y, and we want to define a sheaf on X - we have to take the limit of sections over those open sets which contain $F(U)$.

DEFINITION: Inverse image sheaf, $f^{-1}G$, on X is the sheafification of the presheaf:

$$U \longmapsto \lim_{f(U) \subseteq V} G(V)$$

Sheafification?

Let F be a presheaf on X. There is a sheaf, F^+, and a morphism $\vartheta: F \to F^+$, such that for any sheaf G, and any morphism $\varphi: F \to G$, there is a unique morphism $\Psi: F^+ \to G$, such that:

$$\varphi = \Psi \circ \vartheta$$

$$\begin{tikzcd} F \arrow[r, "\vartheta"] \arrow[d, "\varphi"'] & F^+ \arrow[dl, dashed, "\Psi"] \\ G & \end{tikzcd}$$

F^+ is the **SHEAFIFICATION** of F