

LECTURE FOUR (STALKS, MORPHISMS, AND SHEAFIFIC- ATION)

§ 3.3. STALKS:

IDEA: Take presheaf, $\mathcal{F}$, of abelian groups on a topological space, $X \rightarrow$ Associate to each point $x \in X$ an abelian group, $\mathcal{F}_x$, the **STALK**. We take a small neighbourhood around $x \in X$, and the stalk gives us information of the section in this neighbourhood.

HOW?: Look at sections of open sets which contain $x \in X$ that agree on intersection - requires idea of **DIRECT LIMIT**

The **STALK**, $\mathcal{F}_x$ of $\mathcal{F}$ at $x \in X$, is the **DIRECT LIMIT** of $\mathcal{F}(U)$ for all $U \subseteq X$ containing x .

$$\mathcal{F}_x := \lim_{x \in U} \mathcal{F}(U) := \{ (u, \varphi \in \mathcal{F}(u)) \} / \sim$$

$\sim := (u, \varphi \in \mathcal{F}(u)) \sim (v, g \in \mathcal{F}(v))$ If there exists an open set $W \subseteq u \cap v$ which contains x , such that:

$$\varphi|_W = g|_W$$

REMARK: $\sim$ is reflexive, symmetric, and transitive

If $(s, u) \sim (s, u')$ and $(s', u') \sim (s'', u'')$, one can find open neighbourhoods $V \subset u \cap u'$ and $V' \subset u' \cap u''$ of x , over which s and s' , and s' and s'' , coincide. s and s'' coincide on $V' \cap V$

DEFINITION: The **STALK**, $\overline{F}_x$, at $x \in X$ is the set of equivalence classes:

$$\overline{F}_x = \bigsqcup_{x \in u} F(u) / \sim$$

An element $s_x \in \overline{F}_x$ can be represented by a section $s \in F(u)$ for $x \in u$, and s_x is the germ of the section s at x .

In other words, the germ of s at x is the equivalence class, where for any neighbourhood u of x , there is a natural map $F(u) \rightarrow \overline{F}_x$, which sends the section s to the equivalence class, where (s, u) belongs

GLAZING PRINCIPLE:

- The germ, s_x , of a section, s , vanishes
 $\Leftrightarrow$ The section vanishes on some neighbourhood
of point x
- All elements of $\mathcal{F}_x$ are germs
- $\mathcal{F}$ is the zero sheaf $\Leftrightarrow$ All stalks zero

§3.4. MORPHISMS BETWEEN SHEAVES AND (PRE) SHEAVES:

DEFINITION: Let X be a topological space, and let F and G be (pre) sheaves on X . A

MORPHISM, $\varphi: F \rightarrow G$, consists of morphisms:

$$\varphi(u): F(u) \rightarrow G(u)$$

For each open set, such that, if $V \subseteq U$, then:

$$\begin{array}{ccc} F(u) & \xrightarrow{\varphi(u)} & G(u) \\ \rho_{uv} \downarrow & & \downarrow \rho_{uv} \\ F(v) & \xrightarrow{\varphi(v)} & G(v) \end{array}$$

Commutates.

REMARK: Isomorphic if there exists an inverse to φ , such that:

$$\varphi \circ \varphi^{-1} = \mathbb{1}_G, \quad \varphi^{-1} \circ \varphi = \mathbb{1}_F$$

REMARK: (Pre)sheaves of abelian groups on X , with their morphisms forms a CATEGORY.
Also, sheaves of abelian groups forms a category

§3.5. PUSHFORWARD OF A SHEAF:

Let X be a topological space, F a sheaf on X , and $f: X \rightarrow Y$ a continuous map. One may define a sheaf on Y , $f_* F$ on Y by:

$$(f_* F)(U) = F(f^{-1}U)$$

The restriction maps come from F

One calls $f_* F$ a PUSHFORWARD

EXAMPLE: $p \in Y$ Fixed point

$f: X \rightarrow Y$ Constant map. I.E. $f(x) = p \quad \forall x \in X$

Let F be sheaf on X

For $U \subseteq Y$, one has:

$$\phi_* F(U) = \begin{cases} F(\pi^{-1}(U)) & \text{if } \pi^{-1}(U) \text{ is open} \\ 0 & \text{if } \pi^{-1}(U) \text{ is not open} \end{cases}$$

If one is interested in the scenario where G is a sheaf on Y , and we want to define a sheaf on X - we have to take the limit of sections over those open sets which contain $F(U)$.

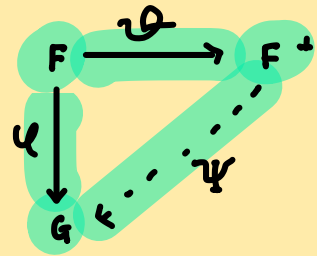
DEFINITION: Inverse image sheaf, $\phi^{-1}G$, on X is the sheafification of the presheaf:

$$U \mapsto \lim_{\phi(U) \subseteq V} G(V)$$

Sheafification?

Let F be a presheaf on X . There is a sheaf, F^+ , and a morphism $\vartheta: F \rightarrow F^+$, such that for any sheaf G , and any morphism $\varphi: F \rightarrow G$, there is a unique morphism $\psi: F^+ \rightarrow G$, such that:

$$\varphi = \psi \circ \vartheta$$



F^+ is the SHEAFIFICATION of F