LECTURE ONE – PREREQUISITES FOR SCHEMES

§ 1. THE PREQUEL:

ON: Algebraic sets, Nullstellensatz, irreducibility, coordinate rings, dimension, regular functions, and morphisms

In CAG we had to consider algebraically closed fields. The studies began by considering points in affine n-space which were all common roots for a set of functions from a coordinate ring.

DEFINITION: Let $S \subset k[x_1, \ldots, x_n]$. Its ZERO SET is:

$$Z(S) = \{x \in \mathbb{A}^n(k) \mid f(x) = 0 \quad \forall f \in S\}$$

An ALGEBRAIC SET is some subset of $\mathbb{A}^n(k)$, which is of this form.

We also had $Z(S) = Z(a)$, for a as the ideal generated by S

Different ideals can define the same algebraic set, thus the radical is introduced.

DEFINITION: The **RADICAL**, $\sqrt{\mathfrak{a}}$ of $\mathfrak{a}$, is :

$$\sqrt{\mathfrak{a}} = \{ f \mid f^r \in \mathfrak{a} \text{ for some } r \in \mathbb{N} \}$$

This helped as $f^r(p) = 0$ $(r > 0)$ if and only if $f(p) = 0$, so $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}})$ $\Rightarrow$ Two ideals with the same radical have the same zero sets.

We can consider the 'converse' as well. Take $V \subset \mathbb{A}^n(k)$. The set of polynomials which vanish at every point of V, $V \subset \mathbb{A}^n(k)$ is:

$$I(V) = \{ f \in k[x_1, \ldots, x_n] \mid f(x) = 0 \ \forall x \in V \}$$

REMARK: $I(V)$ is a radical ideal of $k[x_1, \ldots, x_n]$, and if $W \subset V$, then $I(V) \subset I(W)$

THEOREM (HILBERT'S NULLSTELLENSATZ): Let k be an algebraically closed field. Then the map:

$$\{ \text{Radical ideals } \mathfrak{a} \subset k[x_1, \ldots, x_n] \} \longrightarrow \{ \text{closed sets } X \subset \mathbb{A}^n(k) \}$$

Defined by $\mathfrak{a} \mapsto Z(\mathfrak{a})$ as a bijection, with inverse $X \mapsto I(X)$

THEOREM (HILBERT'S WEAK NULLSTELLENSATZ): Let k be an algebraically closed field. Then:

I.) Every maximal ideal in $k[x_1, \dots, x_n]$ is of the form:

$m = (x_1 - a_1, \dots, x_n - a_n)$

II.) For an ideal $\mathfrak{a}$, the zero set, $Z(\mathfrak{a})$, is empty if and only if $\mathfrak{a} = (1)$

Then we introduced the important idea of IRREDUCIBILITY.

DEFINITION: An algebraic set, $V \subseteq \mathbb{A}^n$, is **REDUCIBLE** if $V = V_1 \cup V_2$ for non-empty sets $V_1, V_2 \subset \mathbb{A}^n$, with $V \neq V_1$ and $V \neq V_2$. Otherwise it is **IRREDUCIBLE.**

A fundamental proposition we have is that an alg. set $X \subset \mathbb{A}^n(k)$ is irreducible if and only if the ideal $I(X)$ is prime. Furthermore, we now can be specific about what is meant by an **AFFINE VARIETY.**

DEFINITION: An **AFFINE VARIETY** is an irreducible algebraic set in $\mathbb{A}^n(k)$.

We then defined a **POLYNOMIAL FUNCTION** on $X \subset \mathbb{A}^n(k)$ as the restriction of some polynomial in $k[x_1, \ldots, x_n]$ to X. Thus, two polynomials, f and g, restrict to the same function on X when their difference, $f-g$, vanishes on X. One may then define the set of polynomial functions on X via the quotient ring:

$$A(X) = k[x_1, \ldots, x_n] \,/\, I(X) \Big\} \text{ COORDINATE RING}$$

$A(X)$ is a tool for transporting geometric properties of X to algebraic properties of $A(X)$ - and vice versa.

ALGEBRA:	GEOMETRY:
Radical ideals of $A(X)$	Closed subsets of X
Prime ideals of $A(X)$	Irreducible algebraic sets in X
Maximal ideals of $A(X)$	Points in X

We can define the dimension of an algebraic set as its dimension as a topological space. I.E. The supremum of the lengths of the chains of distinct irreducible closed subsets of X, where, by definition the chain $X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_n$ has length n.

Our coordinate ring, $A(X)$, of some affine variety X, is an **INTEGRAL DOMAIN**. We thus know there is a fraction field, $k(X)$ - called the field of rational functions on X, and the elements are **RATIONAL FUNCTIONS.**

One classifies $f \in k(X)$ as **REGULAR** at $p \in X$ if one can express it as a fraction $f = a/b$, with $a, b \in A(X)$ and $b(p) \neq 0$.

If $p \in X$, sums and products of regular functions are likewise regular at p.

DEFINITION: The **LOCAL RING** of X at p is the subring of $k(X)$, which contains all elements which are regular at p:

$$\mathcal{O}_{X,p} = \{ f \in k(X) \mid f \text{ is regular at } p \}$$

If X and Y are affine varieties, $U \subset X$ is open, then if $f : U \to Y$ is a continuous map, and $g \in \mathcal{O}_Y(V)$ is a regular function, then the **PULLBACK** of g by f is:

$$f^*(g) = g \circ f \quad \Big\} \text{ Function on open set } f^{-1}(V)$$

Further, $f : U \to Y$ is a **MORPHISM** if it pulls back to regular functions. I.E. $f^*(g) \in \mathcal{O}_X(f^{-1}V)$ for each $V \subset Y$.

$f : U \to Y$ is an **ISOMORPHISM** if it has an inverse map which is also a morphism.