Lecture Six –

Schemes, Morphisms, and closed subschemes

RECAP:

- Why schemes
- Ringed space, locally ringed space
- Pre-sheaf, sheaf
- Morphisms of presheaves & sheaves
- Stalks, direct limits, sheafification
- Abelian schemes, schemes

§ 5.3. MORPHISMS INTO AN AFFINE SCHEME:

THEOREM: Let X be a scheme. Let A be a ring. There is a natural bijection:

$$\mathrm{Hom}_{Sch}(X, \mathrm{Spec}(A)) = \mathrm{Hom}_{Rings}(A, \mathcal{O}_X(X))$$

PROOF: See Ellingsrud & Ottem!

In particular, from this theorem we get that the category of affine schemes is equivalent to the category of schemes

I.E. $A \mapsto \mathrm{Spec}(A)$ Defines functor:

$\mathrm{Spec} : \mathrm{Rings} \to \mathrm{Aff\,Sch}$

Each ring map $\varphi : A \to B$ gets sends to $(f, f^{\#})$, which is a map between ringed spaces

There is, in fact, a functor which takes us in the opposite direction, Γ.

THEOREM: The functors Spec and Γ are, up to equivalence, mutually inverse - giving equivalence between Rings and Aff Sch.

PROOF:

$\Gamma \circ \mathrm{Spec} = \mathbb{1}_{\mathrm{Rings}}$

Given $f : X \to Y$, a map between affine schemes, is induced via unique ring map $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$

Applying Spec takes us back to f

Spec Equivalent to $\mathbb{1}_{\mathrm{Aff\,Sch}}$

§5.4. CLOSED SUBSCHEMES:

U :: Open subset of scheme X

$(U, \mathcal{O}_X|_U)$ Is a scheme! Why?

- Distinguished open set of affine scheme is affine scheme
- This is because if $X = \operatorname{Spec}(R)$, $U = X_f \Rightarrow$ $(U, \mathcal{O}_X|_U) = \operatorname{Spec}(R_f)$
- Distinguished open sets of X contained in U covers U
- So, $(U, \mathcal{O}_X|_U)$ is covered by affine schemes

Let's get specific: If the subset of the scheme is open - we call this an OPEN SUBSCHEME

If the subset is closed, we get a CLOSED SUBSCHEME

Open subschemes → Understood structure.

Closed subscheme → More complicated!

Why? Defining closed subspace of X seems candidate, <u>BUT</u> the sheaf structure not defined in such case.

Instead, use the following idea:

- $X = \mathrm{Spec}(R)$
- For any ideal I we may make closed subset $V(I) \subset X$ into affine scheme by identifying this with $Y = \mathrm{Spec}(R)/I$

↳ Primes of R/I exactly primes of R that contain I modulo I ⟹ $|\mathrm{Spec}(R)/I|$ canonically homeomorphic to closed $V(I) \subset X$

- Closed subscheme of X defined to be scheme Y that is the spectrum of a quotient ring of R

Closed subscheme of $X \longleftrightarrow$ Ideals in R

— x —

How to define for arbitrary scheme?

Thoughts:

- Replace ideal I associated to closed subscheme of affine scheme by a sheaf
- $\mathcal{J} = \mathcal{J}_{Y|X}$ } IDEAL SHEAF

  sheaf of ideals of $\mathcal{O}_X$ given on distinguished open set $V = X_f$ of X by:

  $\mathcal{J}(X_f) = I R_f$

- Structure sheaf, $\mathcal{O}_Y$, identified by:

  $Y = \operatorname{Spec}(R)/I$

Let's formally define.

DEFINITION: Let X be arbitrary scheme.

A **CLOSED SUBSCHEME** Y of X is a closed topological subspace $|Y| \subset |X|$, and a sheaf of rings $\mathcal{O}_Y$, that is a quotient sheaf of the structure sheaf $\mathcal{O}_X$ by a quasicoherent sheaf of ideals, $\mathcal{J}$, such that the intersection of Y with any affine open subset $U \subset X$ is the closed subscheme associated to ideal $\mathcal{J}(U)$

QUASICOHERENT: Quasicoherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ on arbitrary scheme X is sheaf of ideals $\mathcal{J}$ such that, for each open subset U of X, $\mathcal{J}|_U$ is a quasicoherent sheaf of ideals on U.

Take step back: closed subscheme of scheme X is a scheme Z which is embedded as a closed subset $Z \subset X$

Examples:

- $\operatorname{Spec}(k[x]/(x^n))$

  Gives different subschemes of $\mathbb{A}^1_k$

  Underlying topological spaces identical

  I.E. A single point

  Spectra homeomorphic

  Not Isomorphic as schemes - as

  non-isomorphic structure sheaves

- $\mathbb{A}^4_k = \operatorname{Spec}(A)$, $A = k[x_1, x_2, x_3, x_4]$

  Ideals:

  $I_1 = (x_1, x_2)$

  $I_2 = (x_1^2, x_2)$

  $I_3 = (x_1^2, x_1 x_2, x_2^2, x_1 x_4 - x_2 x_3)$

  I_1, I_2, I_3 Have same radical (x_1, x_2) -

  gives rise to same closed subset

  $V(x_1, x_2) \subset \mathbb{A}^4_k$

  BUT Give different closed subschemes

  of $\mathbb{A}^4_k$