LECTURE THREE (SHEAVES)

§3. ON THE TOPIC OF SHEAVES:

One requires an additional piece of data to define the infamous scheme - called sheaves. This represents a frequent concept in maths, where one studies a space by singling in on the local properties.

Three key examples:

- Manifold - locally Euclidean
- complex manifold - locally like $U \subset \mathbb{C}^n$
- Algebraic variety - locally like zero set for set of polynomials

In each respective case, one has a particular set of functions which adequately define them. I.E.:

- Smooth functions (manifolds)
- Holomorphic functions (complex manifolds)
- Polynomials (algebraic variety)

Sheaves are a mechanism for describing such functions.

IDEA: Sheaves satisfy some axioms and these axioms are valid for the examples. Allows us to describe the functions in our examples!

§3.1. PRESHEAVES:

DEFINITION: Let X be a topological space. A **PRESHEAF** of abelian groups, $\mathcal{F}$ on X, consists of the following data:

I.) For each open set $U \subset X$, one has an abelian group, $\mathcal{F}(U)$

II.) For each pair of nested open sets, $V \subset U$, there is a map of groups:

$$\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$$

Said maps are RESTRICTION MAPS, which must satisfy:

III.) For any open set $U \subset X$, $\rho_{UU} = \mathrm{Id}_{\mathcal{F}(U)}$

IV.) For nested open sets $W \subset V \subset U$, $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\rho_{UV}"] \arrow[dr, "\rho_{UW}"'] & \mathcal{F}(V) \arrow[d, "\rho_{VW}"] \\
& \mathcal{F}(W)
\end{tikzcd}$$

COMMUTES!

Call the elements of $\mathcal{F}(U)$ SECTIONS

REMARK: It does not just need to be abelian groups one considers presheaves of. We may also consider:

- Sets
- Rings
- Vector spaces

$\vdots$

§ 3.2. SHEAVES:

We may now specify when exactly a presheaf is a sheaf. In particular, a sheaf is a presheaf where the sections are determined by local data.

DEFINITION: A presheaf is a **SHEAF** if it satisfies:

I.) LOCALITY: Let $U \subset X$ be an open set with an open covering $\mathcal{U} = \{U_i\}_{i \in I}$.

If $s, t \in \mathcal{F}(U)$ are sections, such that:

$$s\big|_{U_i} = t\big|_{U_i} \quad \forall i \quad \Rightarrow s = t$$

II.) GLUING: If U and $\mathcal{U}$ satisfy I.), and $s_i \in \mathcal{F}(U_i)$ is a collection of sections satisfying:

$$s_i\big|_{U_i \cap U_j} = s_j\big|_{U_i \cap U_j} \quad \forall i, j \in I$$

Then there exists a section $s \in \mathcal{F}(U)$, such that

$s|_{U_i} = s_i \quad \forall i$

REMARKS:

- Locality $\Rightarrow$ Sections uniquely determined from their restrictions to smaller open sets
- Gluing $\Rightarrow$ Allowed to patch local sections to form a global one, PROVIDED they agree on overlaps

EXAMPLES:

- Regular functions on an affine variety

$X \subseteq \mathbb{A}^n$ Affine variety defined by $X = V(I)$ for I a radical ideal

For each $U \subseteq X$, let $\mathcal{O}(U)$ be the ring of regular functions, $U \to k$

For $V \subseteq U$, one has the restriction map:

$\rho_{UV} : \mathcal{O}(U) \to \mathcal{O}(V)$

This is the sheaf of regular functions

- **Constant functions**

Let X be a topological space

Define a presheaf $\mathcal{F}$ of constant functions with values in $\mathbb{Z}$

For non-empty open set $U \subseteq X$, set:

$\mathcal{F}(U) = \mathbb{Z}$

For the empty set:

$\mathcal{F}(\emptyset) = 0$

Our restriction maps are identity maps, as constant functions remain constant under restriction

Explicitly:

$\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$, $\rho_{UV}(n) = n$

Let's check axioms!

- **LOCALITY :** Holds as constants that match on each open set have to coincide everywhere ✓

- **GLUING AXIOMS** - Two open sets $U_1, U_2 \subseteq X$ with empty intersection causes issue!

Take $s_1 \in \mathcal{F}(U_1) = 1$, $s_2 \in \mathcal{F}(U_2) = 0$

Since $U_1 \cap U_2 = \emptyset$, the sections trivially agree on empty intersection. I.E.

$s_1|_{U_1 \cap U_2} = 0$, $s_2|_{U_1 \cap U_2} = 0$

No single global constant section $s \in \mathcal{F}(U_1 \cup U_2)$ which restricts to 1 on U_1 and 0 on U_2

FAILS GLUING AXIOM!

Define new sheaf to fix:

$\mathcal{F}'(\emptyset) = 0$

$\mathcal{F}'(U) = \{ f : U \to \mathbb{Z} \mid f(x) = f(y)$ when x, y lie on same connected component of $U \}$

A section then assigns a possible different integer to each connected component of U. I.E.

- For $V \subseteq U$, $\rho_{UV} : \mathcal{F}'(U) \to \mathcal{F}'(V)$, $f \mapsto f|_V$
- For U with connected components indexed by set I:

$$\mathcal{F}'(U) \cong \prod_{i \in I} \mathbb{Z}$$ ✓