§2. SPEC OF A RING:

§2.1. BASICS:

In CAG we had zero sets $\longleftrightarrow$ Maximal ideals

Can prime ideals in general tell us about geom.?

DEFINITION: The SPECTRUM OF A, Spec(A), is the set of all prime ideals in A:

$$\mathrm{Spec}(A) = \{ p \mid p \subseteq A \text{ is a prime ideal} \}$$

'Prime ideals are the points of Spec(A)'

Let's enjoy some nice examples.

EXAMPLES:

- $\mathrm{Spec}(\mathbb{Z})$

  Prime ideals in $\mathbb{Z}$ are generated by the prime numbers and the zero ideal.

  $\mathrm{Spec}(\mathbb{Z}) = \{(0), (2), (3), (5), (7), \ldots\}$

- $\mathrm{Spec}(k[x])$

Prime ideals in $k[x]$ are the zero ideal and the ideal generated by irreducible polynomials.

So, E.G if $k = \mathbb{C}$, the maximal ideals are exactly $(x-a)$, corresponding to $a \in k$. And, (0) corresponds to the affine line.

§2.2. ZARISKI TOPOLOGY:

Indeed, we can define closed sets nicely in the prime spectra, bestowing Spec(A) with its own natural topology - the ZARISKI TOPOLOGY

DEFINITION: Given any subset $I \subseteq A$, the associated vanishing set, $V(I)$, is defined as:

$$V(I) = \{p \in \mathrm{Spec}(A) \mid I \subseteq p\}$$

The closed sets are those prime ideals which contain a particular ideal.

This mirrors the classical Zariski topology, which was $V(I) = \{x \in k^n \mid f(x) = 0 \quad \forall f \in I\}$

That is, in CAG we had points which vanish at polynomials - but now we have prime ideals which contains the polynomials, so vanishing now means to be contained in.

Some basic properties demonstrate that this is a topology.

LEMMA:

I.) For any collection of ideals, $\{a_i\}_{i \in I}$ in A:

$$\bigcap_{i \in I} V(a_i) = V\left(\sum_{i \in I} a_i\right)$$

II.) For two ideals a and b:

$$V(a) \cup V(b) = V(a \cap b) = V(a \cdot b)$$

III.) $V(A) = \emptyset$

IV.) $V(0) = \mathrm{Spec}(A)$

PROOF:

I.) Take $x \in \mathrm{Spec}(A)$. Then $x \in \bigcap_{i \in I} V(a_i)$ if and only if $f(x) = 0$ for all $f \in a_i$, $i \in I$

$\Rightarrow f(x) = 0 \quad \forall f \in \bigcup_{i \in I} a_i$

As $\sum_{i \in I} a_i$ is the ideal generated by $\bigcup_{i \in I} a_i$, it follows that $\bigcap_{i \in I} V(a_i) = V(\sum_{i \in I} a_i)$

II.) $a \cdot b \subset a \cap b$

$\Rightarrow V(a \cap b) \subset V(a \cdot b)$, $V(a) \cup V(b) \subset V(a \cap b)$

we need to verify $V(a \cdot b) \subset V(a) \cup V(b)$.

If $x \notin V(a) \cup V(b)$, then there exists $f \in a$ and $g \in b$ such that $f(x) \neq 0$, $g(x) \neq 0$

$\Rightarrow (fg)(x) = f(x) g(x) \neq 0$ In $K(x)$

$\Rightarrow x \notin V(a \cdot b)$

$\Rightarrow V(a \cdot b) \subset V(a) \cup V(b)$

III.) All prime ideals are proper ideals, so $V(A) = \emptyset$

IV.) Zero ideal (0) is contained in every ideal, so $V(0) = \operatorname{Spec}(A)$ $\square$

LEMMA: For ideals a and b in A:

I.) $V(a) = V(\sqrt{a})$

II.) $V(a) \subset V(b) \iff a = A$

III.) $V(\mathfrak{a}) = \emptyset \iff \mathfrak{a} = A$

IV.) $V(\mathfrak{a}) = \operatorname{Spec}(A) \iff \mathfrak{a} \subset \sqrt{(0)}$

PROOF:

I.) Identity for radical of an ideal is:

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a} \subset \mathfrak{p}} \mathfrak{p}$$

$\Rightarrow$ $\mathfrak{a}$ And $\sqrt{\mathfrak{a}}$ are contained in same prime ideals.

$\Rightarrow V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$

II.) If $V(\mathfrak{a}) \subset V(\mathfrak{b})$ then:

$$\sqrt{\mathfrak{b}} = \bigcap_{\mathfrak{p} \in V(\mathfrak{b})} \mathfrak{p} \subset \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p} = \sqrt{\mathfrak{a}}$$

If $\sqrt{\mathfrak{b}} \subset \sqrt{\mathfrak{a}}$, any prime ideal which contains $\sqrt{\mathfrak{a}}$ auto. contains $\sqrt{\mathfrak{b}}$, so $V(\mathfrak{a}) \subset V(\mathfrak{b})$

III.) By lemma on Zariski topology one has III.), as $V(\mathfrak{a}) = V(1) = \emptyset \iff \sqrt{\mathfrak{a}} = (1)$, which happens if and only if $\mathfrak{a} = (1)$.

IV.) $V(\mathfrak{a}) = V(0) \iff \mathfrak{a} \subset \sqrt{(0)}$

§2.3. RESIDUE FIELDS:

In classical algebraic geometry, functions which we defined on an affine variety took on values in a single field, k

Not anymore! Now each prime ideal, p, comes naturally equipped with its own particular field, which we call the residue field.

DEFINITION: For $p \in A$, the **RESIDUE FIELD**, $k(p)$, is the field of fractions of the integral domain A/p.

Given a specific $f \in A$, one can evaluate this at the prime ideal p. We get:

$$f(p) = f \text{ Mod } p \in A/p \subseteq K(p)$$

§2.4. HILBERT'S NULLSTELLENSATZ ANALOGUE:

Important proposition as it leads to something similar to the Nullstellensatz.

PROPOSITION: For any $S \subset \operatorname{Spec}(A)$:

$V(I(S)) = \overline{S}$

The closure $\overline{\{p\}}$ of the one-point set $\{p\} = V(p)$

PROOF: See Eisenbud and Odden.

COROLLARY: Let A be a ring.

The map:

$\{\text{Radical ideal } a \subset A\} \longleftrightarrow \{\text{Closed sets } W \subset \operatorname{Spec}(A)\}$

Defined by $a \mapsto V(a)$ is a bijection, with inverse $V \mapsto I(W)$

PROOF: See Eisenbud and Odden.

A final note to make in this lecture is on generic points.

DEFINITION: A point x in a closed subset Z of a topological space X is a **GENERIC POINT** for Z if $\overline{\{x\}} = Z$

That is, we have a generic point as a single algebraic 'point' to represent an entire irreducible subvariety.